

# Hall Polynomials via Automorphisms of Short Exact Sequences

By

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Dedicated to Wolfgang Zimmermann

*Abstract.* We present a sum-product formula for the classical Hall polynomial which is based on tableaux that have been introduced by T. Klein in 1969. In the formula, each summand corresponds to a Klein tableau, while the product is taken over the cardinalities of automorphism groups of short exact sequences which are derived from the tableau. For each such sequence, one can read off from the tableau the summands in an indecomposable decomposition, and the size of their homomorphism and automorphism groups. Klein tableaux are refinements of Littlewood-Richardson tableaux in the sense that each entry  $\ell \geq 2$  carries a subscript  $r$ . We describe module theoretic and categorical properties shared by short exact sequences which have the same symbol  $\ell_r$  in a given row in their Klein tableau. Moreover, we determine the interval in the Auslander-Reiten quiver in which indecomposable sequences of  $p^n$ -bounded groups which carry such a symbol occur.

The short exact sequences  $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  of finite abelian  $p$ -groups form the objects in the category  $\mathcal{S}$ ; morphisms are the commutative diagrams. *Prototypes* or *Klein tableaux*, as we call them in this paper, were introduced in [5] as an isomorphism invariant for the objects in  $\mathcal{S}$ . This invariant is finer than the partition triple  $(\alpha, \beta, \gamma)$  which consists of the types of the groups  $A, B, C$ ; and even finer than the Littlewood-Richardson (LR-) tableau associated with  $E$ . More precisely, the Klein tableau has the same entries as the LR-tableau, but each entry bigger than one carries a subscript. We recall that the largest entry is the exponent  $e$  of  $A$ .

It turns out that the Klein tableaux with entries at most two are in one-to-one correspondence with the isomorphism types of short exact

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sequences with first term  $p^2$ -bounded (Proposition 2). For an arbitrary sequence  $E$  we denote the corresponding Klein tableau by  $\Pi(E)$ , and for a Klein tableau  $\Pi$  with entries at most 2 the corresponding sequence by  $E(\Pi, p)$ . One can not expect that Klein tableaux classify arbitrary short exact sequences up to isomorphism since even for  $e = 3$  there occur parametrized families of pairwise non-isomorphic sequences which have the same Klein tableau [9].

In fact, the combinatorial data contained in the Klein tableau  $\Pi$  corresponding to  $E$  describe exactly the isomorphism types of the sequences

$$E|_2^\ell : \quad 0 \rightarrow p^{\ell-2}A/p^\ell A \rightarrow B/p^\ell A \rightarrow B/p^{\ell-2}A \rightarrow 0$$

(and hence also of the sequences

$$E|_1^\ell : \quad 0 \rightarrow p^{\ell-1}A/p^\ell A \rightarrow B/p^\ell A \rightarrow B/p^{\ell-1}A \rightarrow 0).$$

We will see that the tableaux  $\Pi|_i^\ell = \Pi(E|_i^\ell)$  corresponding to these sequences are obtained from  $\Pi$  as suitable restrictions ( $\ell \geq 0, i = 1, 2$ ).

Given finite abelian  $p$ -groups  $A, B, C$  of type  $\alpha, \beta, \gamma$ , respectively, the classical Hall polynomial  $g_{\alpha, \gamma}^\beta(p)$  counts the number of subgroups  $U$  of  $B$  such that  $U \cong A$  and  $B/U \cong C$ . Often, Hall polynomials are computed using LR-tableaux, see for example [6], but in earlier articles, the computation is based on Klein tableaux [5].

In the sum-product formula for Hall polynomials presented in this paper, Klein tableaux control the counting process: The sum is indexed by all Klein tableaux of the given type, and in each summand all the short exact sequences are determined uniquely, up to isomorphism, by suitable restrictions of the corresponding tableau.

**THEOREM 1.** *For partitions  $\alpha, \beta, \gamma$ , the classical Hall polynomial can be computed as*

$$g_{\alpha, \gamma}^\beta(p) = \sum_{\Pi} \prod_{\ell=2}^{e+1} \frac{\#\text{Aut}_S E(\Pi|_1^\ell; p)}{\#\text{Aut}_S E(\Pi|_2^\ell; p)}$$

where the sum is taken over all Klein tableaux  $\Pi$  of type  $(\alpha, \beta, \gamma)$  and  $e = \alpha_1$  is the exponent of the subgroup.

Each Klein tableau can be realized by short exact sequences; if the sequence  $E$  has tableau  $\Pi$  then the summand corresponding to  $\Pi$  in the above formula can be written as

$$g(\Pi; p) = \prod_{\ell=2}^{e+1} \frac{\#\text{Aut}_S E|_1^\ell}{\#\text{Aut}_S E|_2^\ell}.$$

This number counts the subgroups  $U$  of  $B$  such that the sequence  $0 \rightarrow U \rightarrow B \rightarrow B/U \rightarrow 0$  has Klein tableau  $\Pi$ .

We describe module theoretic and categorical properties of the sequences corresponding to a given Klein tableau. We will see how the tableaux determine the size of certain homomorphism and automorphism groups, in particular the size of the groups  $\text{Aut}_{\mathcal{S}} E|_i^\ell$ ,  $i = 1, 2$ , which occur in the formula.

Denote by  $\mathcal{S}_2$  the full subcategory of  $\mathcal{S}$  consisting of short exact sequences  $E$  with  $p^2 A = 0$ . The indecomposable objects in  $\mathcal{S}_2$  are either *pickets*, i.e. sequences with cyclic middle term of the form

$$P_\ell^m : \quad 0 \rightarrow (p^{m-\ell}) \rightarrow \mathbb{Z}/(p^m) \rightarrow \mathbb{Z}/(p^{m-\ell}) \rightarrow 0$$

where  $\ell \leq \min\{m, 2\}$ . Otherwise, they are *bipickets*; here the inclusion is a diagonal embedding of  $\mathbb{Z}/(p^2)$  in a direct sum of two cyclic  $p$ -groups.

$$T_2^{m,r} : \\ 0 \rightarrow ((p^{m-2}, p^{r-1})) \rightarrow \mathbb{Z}/(p^m) \oplus \mathbb{Z}/(p^r) \rightarrow \mathbb{Z}/(p^{m-1}) \oplus \mathbb{Z}/(p^{r-1}) \rightarrow 0$$

where  $1 \leq r \leq m-2$ . To unify notation, we put  $T_2^{m,m-1} = P_2^m$ .

Each object  $T_2^{m,r}$  (where  $1 \leq r \leq m-1$ ,  $(m, r) \neq (2, 1)$ ) occurs as end term of an Auslander-Reiten sequence in  $\mathcal{S}_2$ ,

$$\mathcal{A}^{m,r} : \quad 0 \rightarrow X^{m,r} \rightarrow Y^{m,r} \xrightarrow{v_2^{m,r}} T_2^{m,r} \rightarrow 0;$$

the remaining object  $T_2^{2,1} = P_2^2$  is a projective object in  $\mathcal{S}_2$ .

Consider the lifting functor  $\uparrow^i$  which maps a short exact sequence  $E : 0 \rightarrow A \rightarrow^f B \rightarrow C \rightarrow 0$  in  $\mathcal{S}$  to

$$E\uparrow^i : \quad 0 \rightarrow p^{-i}f(A) \subset B \rightarrow B/p^{-i}f(A) \rightarrow 0$$

where  $p^{-i}f(A) = \{b \in B : p^i b \in f(A)\}$ .

For  $0 \leq i \leq r-1 \leq m-2$ , the liftings  $\mathcal{A}^{m,r}\uparrow^i$  are short exact sequences, unless when  $m = i+2, r = i+1$  in which case  $T_2^{m,r}\uparrow^i = P_m^m$  is a projective object in  $\mathcal{S}_m$ .

Suppose that the Klein tableau  $\Pi$  represents a short exact sequence  $E$ . In our second theorem we interpret the entries in  $\Pi$  in terms of the module structure of  $E$ , and in terms of homological properties of  $E$  as an object in the category  $\mathcal{S}$ . Thus, the combinatorial data defining a Klein tableau have a precise algebraic interpretation within the category  $\mathcal{S}$  of short exact sequences. In this sense,  $\mathcal{S}$  provides a categorification for Klein tableaux.

**THEOREM 2.** *For a short exact sequence  $E \in \mathcal{S}$  with Klein tableau  $\Pi$  and natural numbers  $\ell, m, r$  with  $2 \leq \ell \leq r+1 \leq m$  the following numbers are equal.*

- (1) *The number of boxes  $\boxed{\ell_r}$  in the  $m$ -th row of  $\Pi$ .*
- (2) *The multiplicity of  $T_2^{m,r}$  as a direct summand of  $E|_2^\ell$ .*
- (3) *The  $\mathbb{Z}/(p)$ -dimension of*

$$\frac{\text{Hom}_{\mathcal{S}}(E, T_2^{m,r} \uparrow^{\ell-2})}{\text{Im Hom}_{\mathcal{S}}(E, v_2^{m,r} \uparrow^{\ell-2})}.$$

The corresponding result for LR-tableaux is [10, Theorem 1] where the entries in the tableau are characterized in terms of the picket decomposition of the sequences  $E|_1^\ell$ , and in terms of spaces of homomorphisms from  $E$  into pickets.

As a consequence of Theorem 2, we determine in Corollary 4 the size of the homomorphism groups of the form  $\text{Hom}_{\mathcal{S}}(E, T_2^{m,r})$ .

In Theorem 5 we describe how all the sequences  $E$  with Klein tableau containing a symbol  $\boxed{\ell_r}$  in the  $m$ -th row can be detected within the category  $\mathcal{S}(n)$  of short exact sequences of  $p^n$ -bounded finite abelian groups. Let  $Z = T_2^{m,r} \uparrow^{\ell-2}$  be as in the theorem. We specify an object  $C$  depending only on  $Z$  and  $n$  such that for each sequence  $E$ , the Klein tableau has a symbol  $\boxed{\ell_r}$  in the  $m$ -th row if and only if there are maps  $f : C \rightarrow E$ ,  $g : E \rightarrow Z$  with the property that the composition  $gf$  does not factor through the sink map for  $Z$  in  $\mathcal{S}_2 \uparrow^{\ell-2}$ . In this sense, the sequences  $E$  which have a symbol  $\boxed{\ell_r}$  in the  $m$ -th row of their LR-tableau “lie between”  $C$  and  $Z$ .

We describe the contents of the sections in this paper.

As a gentle introduction to Klein tableaux, we will review in Section 1 combinatorial isomorphism invariants for short exact sequences. We point out that LR-tableaux and Klein tableaux are local invariants in the sense that they depend only on subfactors of the given sequence where the first term is  $p^2$ -bounded.

In Section 2 we study the category  $\mathcal{S}_2$ . To simplify notation, we consider the objects as embeddings ( $A \subset B$ ) of finite abelian  $p$ -groups where  $A$  is  $p^2$ -bounded. We show that Klein tableaux determine the decomposition of arbitrary embeddings in  $\mathcal{S}_2$  as direct sums of pickets and bipickets (Proposition 2), and discuss the Auslander-Reiten quiver for this category.

In Section 3 we give the proof of Theorem 1. The two main ingredients are a sum-product formula for Hall polynomials in [5], and our study

of the action of the group  $\text{Aut}_{\mathbb{Z}}(B)$  on sequences of the form  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ . We illustrate the computations in Theorem 1 in an example; for this we use Corollary 4 to determine the numbers  $\#\text{Aut } E|_i^\ell$ ,  $i = 1, 2$ .

In Section 4 we discuss how Klein tableaux determine the position of short exact sequences within the category  $\mathcal{S}$ . We give the proofs for Theorems 2 and 5, and illustrate both results with examples in the category  $\mathcal{S}(5)$ .

For results and terminology regarding Auslander-Reiten sequences and approximations, we refer the reader to [1] and [2].

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## 1. KLEIN TABLEAUX

In this section we review the following combinatorial isomorphism invariants for short exact sequences:

- Partition triples,
- Littlewood-Richardson tableaux, and
- Klein tableaux.

*Notation.* Let  $R$  be a commutative principal ideal domain,  $p$  a generator of a maximal ideal and  $k = R/(p)$  the residue field. A  $p$ -module is a finite length  $R$ -module which is annihilated by some power of  $p$ .

We denote by  $\mathcal{S}$  the category of all short exact sequences

$$E : \quad 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

of  $p$ -modules, with morphisms given by commutative diagrams. This category is equivalent to the category of embeddings  $E : (A \subset B)$  of  $p$ -modules, with morphisms given by commutative squares. The symbol  $\mathcal{S}$  denotes either one of those categories. For natural numbers  $\ell, n$ , let  $\mathcal{S}_\ell$  and  $\mathcal{S}(n)$  be the full subcategories of  $\mathcal{S}$  of all embeddings  $(A \subset B)$  which satisfy the conditions  $p^\ell A = 0$  and  $p^n B = 0$ , respectively.

**1.1. The partition triple.** We denote the indecomposable  $p$ -module of composition length  $m$  by  $P^m = R/(p^m)$ . It is well known that arbitrary  $p$ -modules are given by partitions:

PROPOSITION 1. *There is a one-to-one correspondence*

$$\{p\text{-modules}\} / \cong \xleftrightarrow{1-1} \{\text{partitions}\}.$$

The partition  $\beta = (\beta_1, \dots, \beta_s)$  corresponds to the  $p$ -module  $M(\beta) = \bigoplus_{i=1}^s P^{\beta_i}$ . Conversely, given a  $p$ -module  $B$ , its type  $\beta = \text{type}(B)$  is obtained via the formula

$$\beta'_i = \dim_k \frac{p^{i-1}B}{p^i B} \quad \text{for } i \in \mathbb{N}$$

where  $\beta'$  is the conjugate of  $\beta$ .

The multiplicity of  $P^m$  in an indecomposable decomposition of  $B$  is

$$\mu_{P^m}(B) = \#\{i \mid \beta_i = m\} = \beta'_m - \beta'_{m+1}.$$

✓

We picture  $P^m$  as a column of  $m$  boxes since the parts of a partition will be given by the lengths of the columns in its diagram.

*Definition.* Given a short exact sequence  $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  of  $p$ -modules, the *partition triple* consists of the three partitions

$$(\text{type}(A), \text{type}(B), \text{type}(C)).$$

Clearly, the partition triple forms an isomorphism invariant for the objects in  $\mathcal{S}$ .

*Example.* In the embedding

$$T_2^{4,2} = (A \subset B) = (((p^2, p)) \subset P^4 \oplus P^2),$$

the submodule  $A$  is cyclic of exponent 2, so  $\alpha = \text{type}(A) = (2)$  and  $\beta = \text{type}(B) = (4, 2)$ . Note that the factor  $B/A$  is not annihilated by  $p^2$ , hence it has type  $\gamma = \text{type}(B/A) = (3, 1)$ ; thus, the bipicket  $T_2^{4,2}$  has partition triple  $((2), (4, 2), (3, 1))$  or  $(2, 42, 31)$  for short.

In general, the partition triple for the picket  $P_\ell^m$  is  $(\ell, m, m - \ell)$ , while the partition triple for the bipicket  $T_2^{m,r}$  is  $((2), (m, r), (m - 1, r - 1))$ .

**1.2. Littlewood-Richardson tableaux.** According to theorems by Green and Klein [4, Section 4], a triple of partitions  $(\alpha, \beta, \gamma)$  can be realized as the partition triple of some embedding  $E \in \mathcal{S}$  if and only if there is an LR-tableau  $\Gamma$  of type  $(\alpha, \beta, \gamma)$ .

*Definition.* A weakly increasing sequence of partitions  $\Gamma = [\gamma^0, \dots, \gamma^e]$  forms a *Littlewood-Richardson tableau (LR-tableau)* provided the following conditions hold:

- (1) For each  $1 \leq \ell \leq e$ , the skew tableau  $\gamma^\ell \setminus \gamma^{\ell-1}$  forms a *horizontal stripe*, that is,  $\gamma_i^\ell - \gamma_i^{\ell-1} \leq 1$  holds for each  $i$ .
- (2) The *lattice permutation property* is satisfied, that is, we have for each  $2 \leq \ell \leq e$  and each  $k \geq 0$ :

$$\sum_{i \geq k} (\gamma_i^\ell - \gamma_i^{\ell-1}) \leq \sum_{i \geq k} (\gamma_i^{\ell-1} - \gamma_i^{\ell-2}).$$

Let  $\alpha$  be the conjugate of the partition defined by the lengths of the horizontal stripes, that is,  $\alpha'_\ell = \sum_i (\gamma_i^\ell - \gamma_i^{\ell-1})$ . Let  $\beta = \gamma^e$  and  $\gamma = \gamma^0$ . Then we say that the LR-tableau has *type*  $(\alpha, \beta, \gamma)$ .

The following observation is immediate:

**LEMMA 1.** *Let  $e \geq 2$ . A weakly increasing sequence  $\Gamma = [\gamma^0, \dots, \gamma^e]$  of partitions has the LR-property if and only if each restriction  $\Gamma|_2^\ell = [\gamma^{\ell-2}, \gamma^{\ell-1}, \gamma^\ell]$  where  $2 \leq \ell \leq e$  does.  $\checkmark$*

We picture the tableau  $\Gamma$  as the diagram  $\beta = \gamma^e$  in which for each  $\ell \geq 1$  the horizontal stripe  $\gamma^\ell \setminus \gamma^{\ell-1}$  is filled with boxes  $\boxed{\ell}$ .

*Example.* The LR-sequence  $\Gamma = [21, 321, 332, 432]$  and its restriction  $\Gamma|_2^3 = [321, 332, 432]$  have the following LR-tableaux.

$$\Gamma : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & 2 \\ \hline 1 & 2 & \\ \hline 3 & & \\ \hline \end{array} \quad \Gamma|_2^3 : \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline 2 & & \\ \hline \end{array}$$

**1.3. The LR-tableau of an embedding.** For an embedding  $E : (A \subset B)$  of  $p$ -modules, the corresponding LR-tableau is obtained as follows. Let  $(\alpha, \beta, \gamma)$  be the partition triple for  $E$  and let  $e = \alpha_1$  be the exponent of  $A$ . The chain of inclusions

$$0 = p^e A \subset p^{e-1} A \subset \dots \subset p^0 A = A$$

yields a chain of epimorphisms

$$B = B/p^e A \twoheadrightarrow B/p^{e-1} A \twoheadrightarrow \dots \twoheadrightarrow B/p^0 A = B/A$$

and hence a weakly decreasing sequence of partitions

$$\beta = \gamma^e \geq \gamma^{e-1} \geq \dots \geq \gamma^0 = \gamma$$

where  $\gamma^i = \text{type}(B/p^i A)$ . Then  $\Gamma = [\gamma^0, \dots, \gamma^e]$  is the LR-tableau for  $E$  [4, Theorem 4.1].

The LR-tableau  $\Gamma$  is an isomorphism invariant for  $E$  refining the partition triple; in fact, the type of  $\Gamma$  is the partition triple  $(\alpha, \beta, \gamma)$  for

$E$ . In [10] we give an interpretation for each entry in  $\Gamma$  in terms of the direct sum decomposition of the subfactors

$$E|_1^\ell = (p^{\ell-1}A/p^\ell A \subset B/p^\ell A) \in \mathcal{S}_1,$$

where  $1 \leq \ell \leq e$ , and in terms of homomorphisms in the category  $\mathcal{S}$ .

*Example.* The LR-tableau for the picket  $P_\ell^m$  is easily computed as  $[m - \ell, m - \ell + 1, \dots, m]$ .

We compute the LR-tableau for the bipicket  $T_2^{42} : (A \subset B)$  from example (1.1); for this we need the types of the factors  $B/p^\ell A$ . From the partition triple we read off that  $\text{type}(B/A) = 31$ . Since

$$(pA \subset B) = (((p^3, 0)) \subset P^4 \oplus P^2)$$

we obtain that  $\text{type}(B/pA) = 32$ . Clearly,  $\text{type}(B/p^2A) = \text{type}(B) = 42$ . So the LR-tableau is  $\Gamma = [31, 32, 42]$ , as pictured below.

We note that both  $P_2^4 \oplus P_0^3 \oplus P_1^2$  and  $T_2^{42} \oplus P_1^3$  have the same LR-tableau  $\Gamma'$ , while their Klein tableaux are different as we will see in (2.2).

$$\Gamma : \begin{array}{|c|c|} \hline & \\ \hline & 1 \\ \hline & \\ \hline 2 & \\ \hline \end{array} \quad \Gamma' : \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline 2 & & \\ \hline \end{array}$$

**1.4. Klein tableaux.** In [5, Section 1] Klein introduces prototypes (which we call Klein tableaux) as refinements of LR-tableaux. Here we use subscript functions for an efficient encoding of the data in the tableau.

*Definition.* Let  $\Gamma = [\gamma^0, \dots, \gamma^e]$  be a weakly increasing sequence of partitions and let  $2 \leq \ell \leq e$ . An  $\ell$ -subscript function is a map

$$\varphi^\ell : \gamma^\ell \setminus \gamma^{\ell-1} \longrightarrow \mathbb{N},$$

defined on the set of boxes in the skew tableau  $\gamma^\ell \setminus \gamma^{\ell-1}$  such that the following conditions are satisfied:

- (i) In each given row, the map  $\varphi^\ell$  is weakly increasing.
- (ii) If a box  $b$  occurs in the  $m$ -th row, then  $\varphi^\ell(b) < m$ .
- (iii) If a box  $b$  lies in the  $m$ -th row, and the box above  $b$  is in  $\gamma^{\ell-1} \setminus \gamma^{\ell-2}$ , then  $\varphi^\ell(b) = m - 1$ .
- (iv) There are at least  $\#(\varphi^\ell)^{-1}(r)$  boxes in the  $r$ -th row of  $\gamma^{\ell-1} \setminus \gamma^{\ell-2}$ .

The data

$$\Pi = [\gamma^0, \dots, \gamma^e; \varphi^2, \dots, \varphi^e]$$

define a *Klein sequence* if  $\Gamma = [\gamma^0, \dots, \gamma^e]$  is an LR-sequence, and if for each  $2 \leq \ell \leq e$  the map  $\varphi^\ell$  is an  $\ell$ -subscript function. We say that the Klein sequence *refines* the LR-sequence; we define its *type* to be the type of the LR-sequence.



The *Klein tableau* represents the data in the Klein sequence as follows. In the LR-tableau  $\Gamma$  replace each box  $b$  with an entry  $\ell \geq 2$  by the symbol  $\boxed{\ell_r}$  where  $r = \varphi^\ell(b)$ . We call  $\ell$  the *entry* or *label* of the box  $b$ ,  $r$  the *subscript*, and  $\boxed{\ell_r}$  the *symbol*. Usually we will omit subscripts that are uniquely determined.

*Remark.* We recall the following equivalent definition for a Klein tableau from [5]. An LR-tableau where each entry  $\ell \geq 2$  carries a subscript is a Klein tableau provided the following conditions are satisfied:

- (i) In any row, the subscripts of the same entry weakly increase from left to right.
- (ii) The subscript of an entry  $\ell \geq 2$  in row  $m$  is at most  $m - 1$ .
- (iii) Any entry  $\ell$  occurring in the same column as an entry  $\ell - 1$  must carry the subscript  $m - 1$  where  $m$  is the row of the entry  $\ell$ .
- (iv) The total number of symbols  $\boxed{\ell_r}$  cannot exceed the number of  $\ell - 1$ 's in row  $r$ .

*Motivation.* Let  $2 \leq \ell \leq s$ . The lattice permutation property in the definition of an LR-tableau makes sure that there is an injective map

$$\psi^\ell : \{\text{boxes labelled } \ell\} \longrightarrow \{\text{boxes labelled } \ell - 1\}$$

such that each  $\psi^\ell(b)$  occurs in some row above  $b$ . The Klein tableau encodes a normalized version of  $\psi^\ell$ , as follows. The map  $\varphi^\ell$  given by  $\varphi^\ell(b) = \text{row}(\psi^\ell(b))$  will satisfy (ii) and (iv). Given that  $\varphi^\ell$  satisfies (ii) and (iv), then there exists a possibly modified version of  $\varphi^\ell$  which will satisfy in addition (iii). In order to make the map weakly increasing (i), compose it with a permutation of boxes in the same row and with the same entry. Note that (iii) will still be satisfied.

*Definition.* For  $\Pi = [\gamma^0, \dots, \gamma^e; \varphi^2, \dots, \varphi^e]$  a Klein tableau and for natural numbers  $u \leq \ell \leq e$  define the *restrictions*

$$\Pi|^\ell = [\gamma^0, \dots, \gamma^\ell; \varphi^2, \dots, \varphi^\ell] \text{ and } \Pi|_u^\ell = [\gamma^{\ell-u}, \dots, \gamma^\ell; \varphi^{\ell-u+2}, \dots, \varphi^\ell].$$

The Klein tableau for the restriction  $\Pi|_u^\ell$  is the skew tableau of shape  $\gamma^\ell \setminus \gamma^{\ell-u}$ ; each of its entries is obtained from the corresponding entry in  $\Pi$  by subtracting  $\ell - u$ . The new entries 1 lose their subscript, while each remaining entry  $x \geq 2$  in  $\Pi|_u^\ell$  inherits its subscript from the corresponding entry  $x + \ell - u$  in  $\Pi$ .

We observe as in Lemma 1:

**LEMMA 2.** *Suppose  $[\gamma^0, \dots, \gamma^e]$  is a weakly increasing sequence of partitions where  $e \geq 2$ , and there are maps  $\varphi^\ell : \gamma^\ell \setminus \gamma^{\ell-1} \rightarrow \mathbb{N}$  for each  $2 \leq \ell \leq e$ . Then the system  $\Pi = [\gamma^0, \dots, \gamma^e; \varphi^2, \dots, \varphi^e]$  is a Klein*

tableau if and only if for each  $2 \leq \ell \leq e$  the restriction

$$\Pi|_2^\ell = [\gamma^{\ell-2}, \gamma^{\ell-1}, \gamma^\ell; \varphi^\ell]$$

is a Klein tableau. ✓

*Example.* For the LR-tableau  $\Gamma$  below, there is only one subscript function  $\varphi^2$  because of condition (iii). However, there are two subscript functions  $\varphi^3$  as the entry 3 can have either subscript 2 or 3. Hence there are the two Klein-tableaux,  $\Pi$  and  $\Pi'$ , which refine  $\Gamma$ .

$$\Gamma : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & 2 \\ \hline 1 & 2 & \\ \hline 3 & & \\ \hline \end{array} \quad \Pi : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & 2_1 \\ \hline 1 & 2_2 & \\ \hline 3_2 & & \\ \hline \end{array} \quad \Pi' : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & 2_1 \\ \hline 1 & 2_2 & \\ \hline 3_3 & & \\ \hline \end{array}$$

**1.5. The Klein tableau of an embedding.** Let  $E : (A \subset B)$  be an embedding, say  $E \in \mathcal{S}(n)$ , with LR-tableau  $\Gamma = [\gamma^0, \dots, \gamma^e]$ . The following partition sequence will define the Klein tableau  $\Pi = \Pi(E)$  corresponding to  $E$  [5, Theorem 2.3].

Given  $\ell \geq 2$ , the chain of submodules

$$\begin{aligned} p^\ell A &= p^\ell A + p(p^{\ell-2}A \cap p^{n-1}B) \subset p^\ell A + p(p^{\ell-2}A \cap p^{n-2}B) \\ &\subset \dots \\ &\subset p^\ell A + p(p^{\ell-2}A \cap B) = p^{\ell-1}A \end{aligned}$$

yields a chain of epimorphisms

$$\begin{aligned} \frac{B}{p^\ell A} &= \frac{B}{p^\ell A + p(p^{\ell-2}A \cap p^{n-1}B)} \twoheadrightarrow \frac{B}{p^\ell A + p(p^{\ell-2}A \cap p^{n-2}B)} \\ &\twoheadrightarrow \dots \\ &\twoheadrightarrow \frac{B}{p^\ell A + p(p^{\ell-2}A \cap B)} = \frac{B}{p^{\ell-1}A} \end{aligned}$$

and hence a weakly decreasing chain of partitions

$$\gamma^\ell = \gamma^{\ell, n-1} \geq \gamma^{\ell, n-2} \geq \dots \geq \gamma^{\ell, 0} = \gamma^{\ell-1}$$

where

$$\gamma^{\ell, r} = \text{type} \left( \frac{B}{p^\ell A + p(p^{\ell-2}A \cap p^r B)} \right).$$

The Klein tableau  $\Pi$  corresponding to  $E$  will have symbols  $\boxed{\ell_r}$  in the skew tableau  $\gamma^{\ell, r} \setminus \gamma^{\ell, r-1}$ , for  $2 \leq \ell$  and  $1 \leq r < n$ , and symbols  $\boxed{1}$  in the skew tableau  $\gamma^1 \setminus \gamma^0$ .

Equivalently, the Klein sequence corresponding to  $E$  is given by the LR-sequence  $\Gamma$  and the  $\ell$ -subscript functions  $\varphi^\ell$  which are defined via

$$\varphi^\ell(b) = r \quad \text{if } b \in \gamma^{\ell, r} \setminus \gamma^{\ell, r-1} \quad (1 \leq r < n, 2 \leq \ell).$$

*Example.* The Klein tableau for a picket  $P_\ell^m$  is determined by condition (iii): The subscript of an entry  $\ell \geq 2$  in row  $m$  is  $m - 1$ .

We compute the Klein tableau for the bipicket  $T_2^{4,2}$  considered in the examples in (1.1) and in (1.3) where we have seen that the LR-tableau  $\Gamma$  for  $T_2^{4,2}$  is as pictured below.

For the Klein tableau it remains to determine the subscript of the entry 2. With  $\ell = 2$ , we have  $p^\ell A = 0$  and  $p^{\ell-2}A = A$ , so the chain of epimorphisms simplifies as

$$B = \frac{B}{p^2 A} \twoheadrightarrow \frac{B}{p(A \cap p^2 B)} \xrightarrow{(*)} \frac{B}{p(A \cap pB)} \twoheadrightarrow \frac{B}{pA}.$$

As  $A \subset pB$  but  $A \not\subset p^2 B$ , the map labelled  $(*)$  is the only proper epimorphism and the partitions representing the above modules are  $42, 42, 32, 32$ . Thus, the subscript is  $r = 2$  and the symbol  $\boxed{2_2}$ .

$$\Gamma : \begin{array}{|c|c|} \hline & \\ \hline & 1 \\ \hline & \\ \hline 2 & \\ \hline \end{array}$$

$$\Pi : \begin{array}{|c|c|} \hline & \\ \hline & 1 \\ \hline & \\ \hline 2_2 & \\ \hline \end{array}$$

In this case, the subscript  $r = 2$  of the entry  $\ell = 2$  is uniquely determined by the tableau as the row of the corresponding entry 1. Hence the subscript can be omitted.

## 2. THE CATEGORY $\mathcal{S}_2$

As a full exact subcategory of  $\mathcal{S}$ , the category  $\mathcal{S}_2$  of all embeddings  $(A \subset B)$  where  $A$  is  $p^2$ -bounded is itself an exact Krull-Remak-Schmidt category, so every object has a unique direct sum decomposition into indecomposables. The indecomposable objects have been determined in [3], they are either pickets or bipickets. We show that arbitrary objects in  $\mathcal{S}_2$  are determined uniquely, up to isomorphism, by their Klein tableaux. We also compute the Auslander-Reiten quiver for  $\mathcal{S}_2$ . It turns out that each indecomposable object is the starting term of a source map; however, some indecomposable objects do not occur as end terms of sink maps.

**2.1. Pickets and Bipickets.** Pickets and bipickets are introduced for  $R$ -modules as for finite abelian groups. It turns out that each indecomposable object in  $\mathcal{S}_2$  is either a picket or a bipicket [3, Theorem 7.5]:

**THEOREM 3.** *The category  $\mathcal{S}_2$  is an exact Krull-Remak-Schmidt category. The indecomposable objects, up to isomorphism, are as follows.*

$$\text{ind } \mathcal{S}_2 = \{P_\ell^m \mid \ell \leq \min\{2, m\}\} \cup \{T_2^{m,r} \mid 1 \leq r \leq m - 2\}.$$

The Klein tableaux of those objects can be computed as in (1.5):

**2.2. Klein tableaux with entries at most 2.** In this section we show that there is a one-to-one correspondence between Klein tableaux with entries at most 2 and isomorphism types of objects in  $\mathcal{S}_2$ .

$$\begin{array}{ccc} \{ \text{objects in } \mathcal{S}_2 \} / \cong & \longleftrightarrow & \{ \text{Klein tableaux with entries at most } 2 \}, \\ E & \longmapsto & \Pi(E). \end{array}$$

$F$	$\mu_F(E)$
$T_2^{m,r}$	$\#\{\text{boxes } \boxed{2_r} \text{ in row } m\} \quad (r < m-1)$
$P_2^m$	$\#\{\text{boxes } \boxed{2_x} \text{ in row } m\} \quad (x = m-1)$
$P_1^m$	$\#\{\text{boxes } \boxed{1} \text{ in row } m\} - \#\{\text{boxes } \boxed{2_m} \text{ in any row}\}$
$P_0^m$	$\#\{\text{empty col. of length } m\} + \#\{\text{boxes } \boxed{\phantom{x}} \text{ in row } m \text{ above } \boxed{2_m}\}$

For the proof of the proposition we use the following

*Proof.* Suppose that  $E = \bigoplus E_i$  is a direct sum of embeddings in  $\mathcal{S}_2$  and that the Klein tableaux for  $E$  and for the  $E_i$  are represented by partition sequences  $\Pi(E) = (\gamma^{\ell,r})$  and  $\Pi(E_i) = (\gamma^{\ell,r}(i))$ . Given  $\ell, r$ , we can recover  $\gamma^{\ell,r}$  as the type of  $F^{\ell,r}(E)$  where  $F^{\ell,r}$  is the functor

$$F^{\ell,r} : \mathcal{S} \rightarrow R\text{-mod}, (A \subset B) \mapsto \frac{B}{p^\ell A + p(p^{\ell-2}A \cap p^r B)}.$$

Since  $F^{\ell,r}$  is additive, the  $m$ -th row of the skew diagram  $\gamma^{\ell,r} \setminus \gamma^{\ell,r-1}$  has length

$$(\gamma^{\ell,r})'_m - (\gamma^{\ell,r-1})'_m = \sum (\gamma^{\ell,r}(i)'_m - \gamma^{\ell,r-1}(i)'_m).$$

Thus the number of symbols  $\boxed{\ell_r}$  in the  $m$ -th row in the Klein tableau  $\Pi(E)$  is obtained by summing up the corresponding numbers in the  $\Pi(E_i)$ .  $\checkmark$

*Proof of the Proposition.* For each indecomposable object in  $\mathcal{S}_2$ , we have computed the Klein tableau, and the Klein tableau for the sum is given by Lemma 3. Hence the map  $E \mapsto \Pi(E)$  is defined. It is onto since each Klein tableau can be realized by an object in  $\mathcal{S}$  [5, Theorem 2.4]; clearly, such an object must be in  $\mathcal{S}_2$ .

It remains to demonstrate for given  $E \in \mathcal{S}_2$  that the multiplicities of the indecomposable summands of  $E$  can be recovered from the Klein tableau  $\Pi = \Pi(E)$ .

Let  $1 \leq r \leq m-2$ . The multiplicity of  $T_2^{m,r}$  as a direct summand of  $E$  equals the number of boxes  $\boxed{2_r}$  in row  $m$ . Namely,  $T_2^{m,r}$  is the only indecomposable object in  $\mathcal{S}_2$  which has this symbol in the given row in its Klein tableau.

The multiplicity of the picket  $P_2^m$  equals the number of boxes  $\boxed{2_x}$  in row  $m$  where  $x = m-1$ .

The multiplicity of pickets of type  $P_1^m$  is given by the number of “unused” boxes  $\boxed{1}$  in row  $m$ . This number is the total number of such boxes, minus the number of symbols  $\boxed{2_m}$  in  $\Pi$ .

Finally, we deal with pickets of type  $P_0^m$ . Together they need to contribute to  $\Pi$  all the empty boxes which have not been obtained otherwise. Given  $m$ , one summand  $P_0^m$  has to be taken for each empty column of length  $m$ , and also for each column of length  $m+1$  which has only a symbol  $\boxed{2_m}$  in row  $m+1$ . (Such columns arise in direct sums like  $P_2^{m+1} \oplus P_0^m$ .)  $\checkmark$

*Example.* We have seen in (1.3) that both  $E : T_2^{42} \oplus P_1^3$  and  $E' : P_2^4 \oplus P_0^3 \oplus P_1^2$  have the same LR-tableau  $\Gamma$ . The Klein tableaux of the indecomposable summands have been determined in (1.5). Lemma 3 yields the Klein tableaux  $\Pi = \Pi(E)$  and  $\Pi' = \Pi(E')$  of the sums:

$$\Gamma' : \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline 2 & & \\ \hline \end{array} \quad \Pi : \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline 2_2 & & \\ \hline \end{array} \quad \Pi' : \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline 2_3 & & \\ \hline \end{array}$$

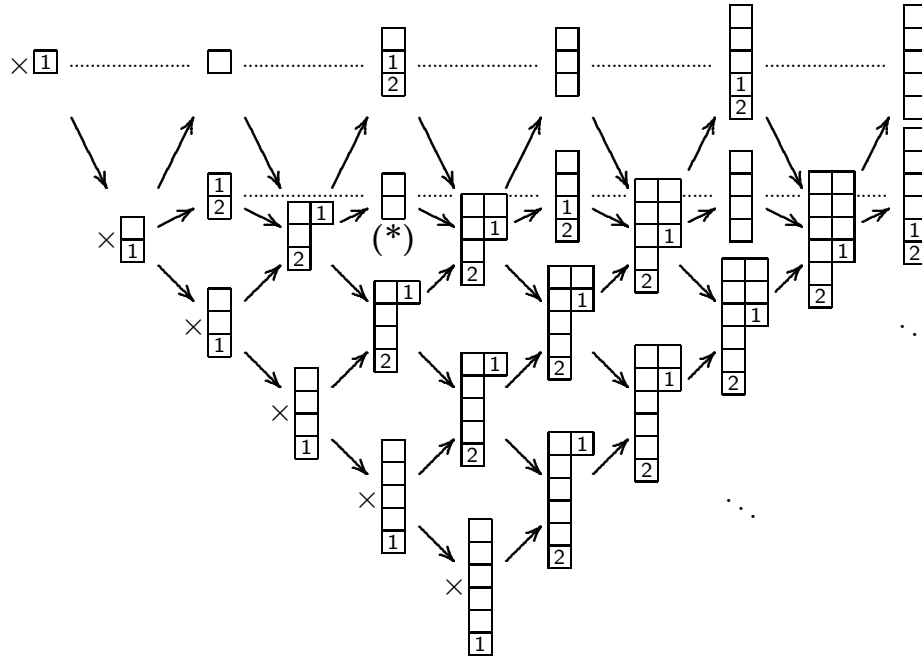
Conversely, using Proposition 2 we can retrieve the direct sum decompositions from the Klein tableaux.

**2.3. Auslander-Reiten sequences in  $\mathcal{S}_2$ .** Let  $\mathcal{S}_2(n) = \mathcal{S}_2 \cap \mathcal{S}(n)$  be the full subcategory of  $\mathcal{S}$  of all pairs  $(A \subset B)$  which satisfy  $p^2 A = 0$  and  $p^n B = 0$ . For each  $n$ , the category  $\mathcal{S}_2(n)$  has Auslander-Reiten sequences [9]. It is the aim of this section to describe the Auslander-Reiten theory for  $\mathcal{S}_2$ .

Let  $\mathcal{E}$  be an Auslander-Reiten sequence in  $\mathcal{S}_2(n)$  with modules in  $\mathcal{S}_2(n-1)$ , and let  $n' \geq n$ . It follows from the description of the Auslander-Reiten quivers in [9] that  $\mathcal{E}$  is also an Auslander-Reiten sequence in  $\mathcal{S}_2(n')$ . Hence, such a sequence  $\mathcal{E}$  is an Auslander-Reiten sequence even in  $\mathcal{S}_2$ .

More precisely, each indecomposable module in  $\mathcal{S}_2$  has a source map in  $\mathcal{S}_2$ , and each indecomposable object not of type  $P_1^m$  has a sink map in  $\mathcal{S}_2$ . The pickets of the form  $P_1^m$  for  $m \geq 1$  are end terms of Auslander-Reiten sequences in each of the categories  $\mathcal{S}_2(n)$  for  $n \geq m$ , but those sequences depend on  $n$ . We label those objects with  $\times$  since they are neither projective, nor do they occur as end terms of sink maps in  $\mathcal{S}_2$ . Finally, the module  $P_2^2$  is an (Ext-) projective object in each of the categories  $\mathcal{S}_2(n)$ , hence also in  $\mathcal{S}_2$ .

Here is the partial Auslander-Reiten quiver for  $\mathcal{S}_2$ ; we picture each object by its Klein tableau:



In the diagram, the sequence ending at  $T_2^{4,2}$  is labelled by (\*). We will visualize in (4.6), Example (2), how this sequence determines all the

indecomposable objects in the Auslander-Reiten quiver for  $\mathcal{S}(5)$  which have a symbol  $\boxed{2_2}$  in the 4-th row of their Klein tableau.

For later use we record the sink maps ending at a picket of type  $P_2^m$ :

$$v_2^m : \begin{cases} P_1^2 \rightarrow P_2^2 & \text{if } m = 2 \\ T_2^{m,m-2} \rightarrow P_2^m & \text{if } m \geq 3 \end{cases}$$

and the sink maps ending at a bipicket of type  $T_2^{m,r}$ :

$$v_2^{m,r} : \begin{cases} P_0^1 \oplus P_2^2 \oplus P_1^3 \rightarrow T_2^{3,1} & \text{if } m = 3, r = 1 \\ P_0^{m-2} \oplus P_2^{m-1} \oplus T_2^{m,m-3} \rightarrow T_2^{m,m-2} & \text{if } m \geq 4, r = m-2 \\ P_1^m \oplus T_2^{m-1,1} \rightarrow T_2^{m,1} & \text{if } m \geq 4, r = 1 \\ T_2^{m,r-1} \oplus T_2^{m-1,r} \rightarrow T_2^{m,r} & \text{if } 2 \leq r \leq m-3 \end{cases}$$

To unify notation we also write  $T_2^{m,m-1} = P_2^m$  and define  $v_2^{m,m-1} = v_2^m$ .

### 3. HALL POLYNOMIALS

We assume throughout this section that the residue field  $k = R/(p)$  is the finite field of  $q$  elements. The sum-product formula in [5], in conjunction with an application of the orbit equation, yields the formula for the Hall polynomial in Theorem 1. In an example we illustrate how Klein tableaux control the counting process.

**3.1. The action of the automorphism group of  $B$ .** Let  $(A \subset B)$  be an embedding in  $\mathcal{S}$ . The group  $G = \text{Aut}_R B$  acts on the set

$$\{(U \subset B) \in \mathcal{S} \mid U \subset B\}$$

via  $\beta \cdot (U \subset B) = (\beta(U) \subset B)$ . The cardinality of the orbit of  $(A \subset B)$  under this action is the Hall multiplicity of the embedding

$$g(A \subset B) = \#\{U \subset B \mid (U \subset B) \cong (A \subset B) \text{ in } \mathcal{S}\}$$

while the stabilizer of  $(A \subset B)$  is the automorphism group  $\text{Aut}_{\mathcal{S}}(A \subset B)$ . From the orbit formula we obtain

$$g(A \subset B) = \frac{\#\text{Aut}_R B}{\#\text{Aut}_{\mathcal{S}}(A \subset B)}.$$

**3.2. Klein's sum-product formula.** We deduce Theorem 1 by applying the above to a sum-product formula which is implicitly in Klein's original article on the computation of Hall polynomials [5]. We first exhibit the formula and then use (3.1) to prove Theorem 1.

*Notation.* Let  $\Pi$  be a Klein tableau of type  $(\alpha, \beta, \gamma)$ . The Hall multiplicity of  $\Pi$  in  $R_p\text{-mod}$  is denoted by

$$g(\Pi; q) = \#\{U \subset M(\beta) \mid \Pi((U \subset M(\beta))) = \Pi\}.$$

PROPOSITION 3. *For partitions  $\alpha, \beta, \gamma$  and  $e = \alpha_1$ , the Hall polynomial in  $R_p$ -mod can be computed as*

$$g_{\alpha\gamma}^\beta(q) = \sum_{\Pi} \prod_{\ell=2}^{e+1} \frac{g(\Pi|_2^\ell; q)}{g(\Pi|_1^\ell; q)}$$

where the sum is taken over all Klein tableaux of type  $(\alpha, \beta, \gamma)$ .

*Proof.* According to [5, Corollaries 1-3, p. 77], the Hall polynomial is computed as

$$(*) \quad g_{\alpha\gamma}^\beta(q) = q^{\|\beta\| - \|\alpha\| - \|\gamma\|} \sum_{\Pi} \prod_{\ell=2}^{e+1} F(\Pi|_2^\ell; \frac{1}{q}),$$

where the sum is over all Klein tableaux of type  $(\alpha, \beta, \gamma)$ . Here,  $F(\Pi, t)$  is a polynomial in  $t$  given explicitly in [5, (3.5)]. The moment of a partition  $\mu$  is given as  $\|\mu\| = \sum_{i \geq 1} \binom{\mu'_i}{2}$ , so in particular  $\|\alpha\| = \sum_{\ell} \|(1^{\alpha'_\ell})\|$ , and we have

$$q^{\|\beta\| - \|\gamma\| - \|\alpha\|} = \prod_{\ell \geq 1} q^{\|\gamma^\ell\| - \|\gamma^{\ell-1}\| - \|(1^{\alpha'_\ell})\|} = \prod_{\ell \geq 2} q^{\|\gamma^{\ell-1}\| - \|\gamma^{\ell-2}\| - \|(1^{\alpha'_{\ell-1}})\|}.$$

Hence the first factor in  $(*)$  can be distributed over the factors in the product. Let  $U \subset B$  be an elementary subgroup such that the embedding  $(U \subset B)$  has Klein tableau  $\Pi|_1^\ell$ . According to [5, Theorem 3.7] the number of subgroups  $A$  in  $B$  which have Klein tableau  $\Pi|_2^\ell$  and for which  $pA = U$  holds is computed as

$$h(\Pi|_2^\ell, q) = q^{\|\gamma^{\ell-1}\| - \|\gamma^{\ell-2}\| - \|(1^v)\|} F(\Pi|_2^\ell; \frac{1}{q})$$

where  $v = |\gamma^{\ell-1} \setminus \gamma^{\ell-2}|$ . Since the number of elementary subgroups  $U$  in  $B$  of tableau  $\Pi|_1^\ell$  is  $g(\Pi|_1^\ell, q)$ , and since all such embeddings  $(U \subset B)$  are isomorphic in  $\mathcal{S}$ , we can write  $h(\Pi|_2^\ell; q)$  as a quotient of two Hall multiplicities, as needed:

$$h(\Pi|_2^\ell, q) = \frac{g(\Pi|_2^\ell, q)}{g(\Pi|_1^\ell, q)}.$$

✓

We deduce the formula in Theorem 1:

*Proof of Theorem 1.* The key point is that each Klein tableau with entries at most 2 determines a unique isomorphism class of short exact sequences, so that we can apply the orbit formula in (3.1). In particular,



we obtain for the embedding  $E(\Pi|_2^\ell; q) = (A \subset B)$  corresponding to the Klein tableau  $\Pi|_2^\ell$ :

$$g(\Pi|_2^\ell; q) = g((A \subset B)) = \frac{\# \text{Aut}_R B}{\# \text{Aut}_S(A \subset B)}$$

Hence we have

$$\frac{g(\Pi|_2^\ell; q)}{g(\Pi|_1^\ell; q)} = \frac{\# \text{Aut}_S(pA \subset B)}{\# \text{Aut}_S(A \subset B)} = \frac{\# \text{Aut}_S E(\Pi|_1^\ell; q)}{\# \text{Aut}_S E(\Pi|_2^\ell; q)}.$$

It follows from Proposition 3 that

$$g_{\alpha\gamma}^\beta(q) = \sum_{\Pi} \prod_{\ell=2}^{e+1} \frac{\# \text{Aut}_S E(\Pi|_1^\ell; q)}{\# \text{Aut}_S E(\Pi|_2^\ell; q)}.$$

✓

### 3.3. Remarks.

- (1) Consider the corresponding sum-product formula for LR-tableaux from [6, II, (4.2) and (4.9)]:

$$g_{\alpha\gamma}^\beta(q) = \sum_{\Gamma} \prod_{\ell=2}^{e+1} h(\Gamma|_2^\ell; q),$$

where the sum is over all LR-tableaux  $\Gamma$  of type  $(\alpha, \beta, \gamma)$ . By  $\Gamma|_i^\ell$  we denote the restriction of the tableau to the entries  $\ell, \ell - 1, \dots, \ell - i$ , as for Klein tableaux. Again, the factors can be written as

$$h(\Gamma|_2^\ell; q) = \frac{g(\Gamma|_2^\ell; q)}{g(\Gamma|_1^\ell; q)},$$

where  $g(\Gamma|_1^\ell; q) = g(\Pi|_1^\ell; q)$  counts elementary embeddings. However, the  $p^2$ -bounded embeddings counted by  $g(\Gamma|_2^\ell; q)$  are not necessarily isomorphic.

- (2) For any LR-tableau  $\Gamma$ , the formula  $g(\Gamma; q) = \sum_{\Pi} g(\Pi; q)$  holds where the sum is over all Klein tableaux  $\Pi$  refining  $\Gamma$ . Note that each of the polynomials  $g(\Gamma; q)$ ,  $g(\Pi; q)$  is monic, so there is a unique Klein tableau  $\Pi_D$ , called in [5] the *dominant Klein tableau corresponding to*  $\Gamma$ , which refines  $\Gamma$  and which is such that  $g(\Pi_D; q)$  has the same degree as  $g(\Gamma; q)$ , and hence the same degree as the Hall polynomial.
- (3) In the above sum-product formula  $g_{\alpha\gamma}^\beta(q) = \sum_{\Pi} \prod_{\ell=2}^{e+1} h(\Pi|_2^\ell; q)$ , the factors are indexed by uniquely determined isomorphism classes of  $p^2$ -bounded embeddings. Note that one cannot expect a refined version of this formula where the factors are described

by uniquely determined isomorphism types of  $p^3$ -bounded embeddings since parametrized families occur already in the case where  $B$  has exponent 7 [9].

**3.4. Example.** In order to demonstrate that the formula in Theorem 1 is computationally feasible, we determine the Hall polynomial  $g_{321,21}^{432}$ . The following lemma will provide us with the size of some homomorphism and automorphism groups.

$$\begin{aligned} \text{LEMMA 4.} \quad (1) \quad & \# \text{End}_{\mathcal{S}} T_2^{m,r} = q^{m+3r-1}, \\ & \# \text{Aut}_{\mathcal{S}} T_2^{m,r} = (1 - \frac{1}{q})q^{m+3r-1} \\ (2) \quad & \# \text{Hom}_{\mathcal{S}}(P_u^v, P_\ell^m) = q^{\min\{m, v - \max\{0, u - \ell\}\}}, \\ & \# \text{Aut}_{\mathcal{S}} P_\ell^m = (1 - \frac{1}{q})q^m \\ (3) \quad & \# \text{Hom}_{\mathcal{S}}(T_2^{v,w}, P_\ell^m) = \begin{cases} q^{\min\{v-1, m\} + \min\{w-1, m\}}, & \text{if } \ell = 0 \\ q^{\min\{v-1, m\} + \min\{w, m\}}, & \text{if } \ell = 1 \\ q^{\min\{v, m\} + \min\{w, m\}}, & \text{if } \ell \geq 2 \end{cases} \\ (4) \quad & \# \text{Hom}_{\mathcal{S}}(P_u^v, T_2^{m,r}) = \begin{cases} q^{\min\{v, r\} + \min\{v, m\}}, & \text{if } u = 0 \\ q^{\min\{v-1, r\} + \min\{v, m\}}, & \text{if } u = 1 \\ q^{\min\{v-u+1, r\} + \min\{v-u+1, m\}}, & \text{if } u \geq 2 \end{cases} \end{aligned}$$

*Proof.* Corollary 4 yields the results stated in (1) and in (4). For (2) and (3), use the equality

$$\text{Hom}_{\mathcal{S}}((A \subset B), P_\ell^m) = \text{Hom}_R(B/p^\ell A, P^m)$$

discussed in [10] and after Corollary 4. ✓

*Example.* For  $\alpha = (321)$ ,  $\beta = (432)$  and  $\gamma = (21)$  as in (1.4) we compute the Hall polynomial  $g_{\alpha, \gamma}^\beta$  using the formula in Theorem 1. There are three Klein tableaux of type  $(\alpha, \beta, \gamma)$ :

$$\Pi_1 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & 2 \\ \hline 1 & 3 & \\ \hline 2 & & \\ \hline \end{array} \quad \Pi = \Pi_2 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & 2 \\ \hline 1 & 2 & \\ \hline 3 & 2 & \\ \hline \end{array} \quad \Pi_3 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & 2 \\ \hline 1 & 2 & \\ \hline 3 & 3 & \\ \hline \end{array}$$

The sum in the formula is indexed by the Klein tableaux of the given type:

$$g_{\alpha, \gamma}^\beta = g(\Pi_1) + g(\Pi_2) + g(\Pi_3)$$

We put  $\Pi = \Pi_2$  and compute  $g(\Pi)$  first:

$$g(\Pi) = \frac{\# \text{Aut } \mathcal{E}(\Pi|_1^2)}{\# \text{Aut } \mathcal{E}(\Pi|_2^2)} \cdot \frac{\# \text{Aut } \mathcal{E}(\Pi|_1^3)}{\# \text{Aut } \mathcal{E}(\Pi|_2^3)} \cdot \frac{\# \text{Aut } \mathcal{E}(\Pi|_1^4)}{\# \text{Aut } \mathcal{E}(\Pi|_2^4)}$$

The restrictions of  $\Pi$ ,

$$\Pi|_2^2 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & 2 \\ \hline 1 & 2 & \\ \hline \end{array} \quad \Pi|_1^2 : \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline \end{array} \quad \Pi|_2^3 : \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline 1 & & \\ \hline 2 & & \\ \hline \end{array} \quad \Pi|_1^3 = \Pi|_2^4 : \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline 1 & & \\ \hline \end{array} \quad \Pi|_1^4 : \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

define the following direct sum decompositions of the short exact sequences corresponding to the tableaux (Proposition 2):

$$g(\Pi) = \frac{\# \operatorname{Aut}(P_0^3 \oplus P_1^3 \oplus P_1^2)}{\# \operatorname{Aut}(P_1^3 \oplus P_2^3 \oplus P_2^2)} \cdot \frac{\# \operatorname{Aut}(P_1^4 \oplus P_0^3 \oplus P_0^2)}{\# \operatorname{Aut}(T_2^{42} \oplus P_1^3)} \cdot \frac{\# \operatorname{Aut}(P_0^4 \oplus P_0^3 \oplus P_0^2)}{\# \operatorname{Aut}(P_1^4 \oplus P_0^3 \oplus P_0^2)}$$

For the first factor on the right hand side, note that the modules in the numerator and in the denominator correspond to each other under the functors  $\uparrow, \downarrow$  defined in (4.1). The adjointness isomorphism in Lemma 5 yields an isomorphism between the automorphism groups. Hence the first factor is 1.

Also the third factor is 1 since the total spaces of the direct sums in numerator and denominator agree while the subgroups are invariant under automorphisms.

We determine the second factor using the lemma above.

$$\begin{aligned} \# \operatorname{Aut}_{\mathcal{S}}(P_1^4 \oplus P_0^3 \oplus P_0^2) &= (1 - \frac{1}{q})^3 q^{23} \\ \# \operatorname{Aut}_{\mathcal{S}}(T_2^{42} \oplus P_1^3) &= (1 - \frac{1}{q})^2 q^{22} \end{aligned}$$

Hence  $g(\Pi_2) = q - 1$ . The computation of  $g(\Pi_1)$  and  $g(\Pi_3)$  turns out to be even less involved since no bipickets occur. The numbers are  $g(\Pi_1) = q^2 = g(\Pi_3)$ .

In conclusion,  $g_{321,21}^{432}(q) = g(\Pi_1) + g(\Pi_2) + g(\Pi_3) = 2q^2 + q - 1$ .

#### 4. HOMOMORPHISMS IN $\mathcal{S}$

In this section we interpret the entries in the Klein tableau in terms of homomorphisms in the category  $\mathcal{S}$ .

**4.1. Lifting and Reducing.** We consider the following two endofunctors on  $\mathcal{S}$ :

$$\begin{aligned} \uparrow : (A \subset B) &\longmapsto (p^{-1}A \subset B) && \text{“lifting”} \\ \downarrow : (A \subset B) &\longmapsto (pA \subset B) && \text{“reducing”} \end{aligned}$$

Here  $p^{-1}A$  denotes the subgroup  $\{b \in B \mid pb \in A\}$ . For  $s$  a nonnegative integer,  $\uparrow^s, \downarrow_s$  are  $s$  iterations of  $\uparrow$  and  $\downarrow$ , respectively.

The following corollary is an immediate consequence of the formulas  $p^{-1}pA = A + \operatorname{soc} B$ ,  $pp^{-1}A = A \cap \operatorname{rad} B$ ,  $pp^{-1}pA = pA$ , and  $p^{-1}pp^{-1}A = p^{-1}A$ .

$$(A \subset B) \uparrow \downarrow \uparrow = (A \subset B) \uparrow \quad and \quad (A \subset B) \downarrow \uparrow \downarrow = (A \subset B) \downarrow.$$
$$P_\ell^{m \uparrow s} = P_u^m \quad \text{where} \quad u = \min\{\ell + s, m\}.$$
$$T_2^{m,r} \uparrow^s = ( ((p^{m-2-s}, p^{r-1-s}), (0, p^{r-s})) \subset P^m \oplus P^r )$$
$$T_2^{m,r} \uparrow^s = P_u^m \oplus P_r^r \quad \text{where} \quad u = \min\{1+s, m\}.$$

Here are some Klein tableaux of raised embeddings.

$$P_2^5: \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad P_2^5 \uparrow^2: \begin{array}{|c|} \hline \square \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \quad T_2^{5,3}: \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & 1 \\ \hline \square & \square \\ \hline 2 & \square \\ \hline \end{array} \quad T_2^{5,3} \uparrow^2: \begin{array}{|c|c|} \hline \square & 1 \\ \hline \square & 2 \\ \hline 1 & 3 \\ \hline 2 & \square \\ \hline 4 & \square \\ \hline \end{array}$$

LEMMA 5. *The functors  $\downarrow_s, \uparrow^s$  form an adjoint pair. More precisely, we have for  $E, F$  in  $\mathcal{S}$ :*

$$\mathrm{Hom}_{\mathcal{S}}(E \downarrow_s, F) = \mathrm{Hom}_{\mathcal{S}}(E, F \uparrow^s).$$

$$\Pi(E|_{\downarrow_s}) = [\gamma^s, \dots, \gamma^e; \varphi^{2+s}, \dots, \varphi^e] = \Pi|_{e-s}^e.$$

*Proof.* Suppose  $E$  is given by the embedding  $(A \subset B)$  and its Klein tableau  $\Pi$  is defined by the partition sequence  $(\gamma^{\ell,r})_{\ell,r}$  where  $\gamma^{\ell,r} = \text{type}(B/p^\ell A + p(p^{\ell-2} A \cap p^r B))$ . Then  $E|_s = (p^s A \subset B)$ , hence the Klein tableau  $\Pi(E|_s)$  is defined by the partition sequence  $(\gamma^{\ell+s,r})_{\ell,r}$ .  $\checkmark$

**4.2. The categories  $\mathcal{S}_2 \uparrow^{\ell-2}$ .** For  $\ell \geq 2$ , the category  $\mathcal{S}_2 \uparrow^{\ell-2}$  consists of all pairs  $(A \subset B)$  in  $\mathcal{S}_\ell$  where  $\text{soc}^{\ell-2} B \subset A$ . For each indecomposable pair not in  $\mathcal{S}_{\ell-1}$  we determine the sink map in  $\mathcal{S}_2 \uparrow^{\ell-2}$  and use this information to picture the partial Auslander-Reiten quiver.

*Definition.* For  $\mathcal{U}$  one of the categories  $\mathcal{S}$ ,  $\mathcal{S}_\ell$ ,  $\mathcal{S}(n)$ , or  $\mathcal{S}_\ell(n)$ , let  $\mathcal{U} \uparrow$  and  $\mathcal{U} \downarrow$  be the full subcategories of  $\mathcal{S}$  of all pairs  $(A \subset B) \uparrow$  and  $(A \subset B) \downarrow$ , respectively, where  $(A \subset B) \in \mathcal{U}$ .

LEMMA 7. *The functors  $\downarrow, \uparrow$  induce categorical isomorphisms*

$$\mathcal{S} \uparrow \cong \mathcal{S} \downarrow, \quad \mathcal{S}(n) \uparrow \cong \mathcal{S}(n) \downarrow, \quad \mathcal{S}_\ell \uparrow \cong \mathcal{S}_{\ell+1} \downarrow, \quad \mathcal{S}_\ell(n) \uparrow \cong \mathcal{S}_{\ell+1}(n) \downarrow.$$

*Proof.* According to Corollary 1, the functors  $\downarrow \uparrow, \uparrow \downarrow$  are the identity on each object and on each homomorphism group.  $\checkmark$

PROPOSITION 4. *Let  $\ell > 2$ . If  $(A \subset B) \uparrow^{\ell-2}$  in  $\mathcal{S}_2 \uparrow^{\ell-2}$  is an indecomposable object with  $p^{\ell-1}A \neq 0$ , and  $v$  the sink map for  $(A \subset B)$  in  $\mathcal{S}_2$ , then the sink map  $v'$  for  $(A \subset B) \uparrow^{\ell-2}$  in  $\mathcal{S}_2 \uparrow^{\ell-2}$  is the minimal version of  $v \uparrow^{\ell-2}$ . In particular, for  $E \in \mathcal{S}$ ,*

$$\text{Im Hom}_{\mathcal{S}}(E, v') = \text{Im Hom}_{\mathcal{S}}(E, v \uparrow^{\ell-2}).$$

*Proof.* By Lemma 7, the category  $\mathcal{S}_2 \uparrow^{\ell-2}$  is equivalent to the full subcategory of  $\mathcal{S}_2$  of all pairs  $(A \subset B)$  where  $A \subset p^{\ell-2}B$ . We obtain the sink maps in the proposition by taking minimal versions of the liftings of the corresponding sink maps in  $\mathcal{S}_2$ . We are interested in pickets of the form  $P_\ell^m$  where  $\ell \leq m$  (cases (a) and (b)), and in bipickets of type  $T_2^{m,r} \uparrow^{\ell-2}$  where  $\ell \leq r+1 < m$  (cases (c) and (d)).

- (a) The picket  $P_\ell^\ell$  is projective in  $\mathcal{S}_\ell$ , and hence in  $\mathcal{S}_2 \uparrow^{\ell-2}$ , and has as sink map the inclusion

$$P_{\ell-1}^\ell \rightarrow P_\ell^\ell.$$

This is the minimal version of the map  $v \uparrow^{\ell-2} : P_{\ell-1}^\ell \oplus P_{\ell-2}^{\ell-2} \rightarrow P_\ell^\ell$ , obtained by lifting the sink map  $v : T_2^{\ell,\ell-2} \rightarrow P_2^\ell$  in  $\mathcal{S}_2$  (see the example in (4.1)).

- (b) For  $m > \ell$ , the Auslander-Reiten sequence for the picket  $P_2^m$  in  $\mathcal{S}_2$ ,

$$0 \rightarrow P_0^{m-2} \rightarrow T_2^{m,m-2} \rightarrow P_2^m \rightarrow 0$$

yields the Auslander-Reiten sequence for the picket  $P_\ell^m$  in  $\mathcal{S}_2 \uparrow^{\ell-2}$ ,

$$0 \rightarrow P_{\ell-2}^{m-2} \rightarrow T_2^{m,m-2} \uparrow^{\ell-2} \rightarrow P_\ell^m \rightarrow 0.$$

- (c<sub>1</sub>) We first consider the case where  $\ell = r + 1 < m - 1$ . The first term of the sink map in  $\mathcal{S}_2$ ,

$$T_2^{m-1,r} \oplus T_2^{m,r-1} \rightarrow T_2^{m,r}$$

decomposes after lifting into  $\mathcal{S}_2 \uparrow^{\ell-2}$ ; the lifted map is a right almost split morphism in this category:

$$T_2^{m-1,r} \uparrow^{\ell-2} \oplus P_{\ell-1}^m \oplus P_{r-1}^{r-1} \rightarrow T_2^{m,r} \uparrow^{\ell-2}$$

The minimal version of this map is the sink map  $v$  for  $T_2^{m,r} \uparrow^{\ell-2}$  in the Auslander-Reiten sequence in  $\mathcal{S}_2 \uparrow^{\ell-2}$ ,

$$0 \rightarrow P_{\ell-1}^{m-1} \rightarrow T_2^{m-1,r} \uparrow^{\ell-2} \oplus P_{\ell-1}^m \xrightarrow{v} T_2^{m,r} \uparrow^{\ell-2} \rightarrow 0.$$

- (c<sub>2</sub>) Similarly in case  $\ell = r + 1 = m - 1$ , one shows that the sink map for  $T_2^{m,r} \uparrow^{\ell-2}$  in  $\mathcal{S}_2 \uparrow^{\ell-2}$  is the epimorphism in the sequence:

$$0 \rightarrow P_{\ell-1}^\ell \rightarrow P_\ell^\ell \oplus P_{\ell-2}^{\ell-1} \oplus P_{\ell-1}^{\ell+1} \xrightarrow{v} T_2^{\ell+1,\ell-1} \uparrow^{\ell-2} \rightarrow 0.$$

- (d) Finally for  $\ell < r + 1 < m$ , the Auslander-Reiten sequence in  $\mathcal{S}_2$  ending in  $T_2^{m,r}$  yields an Auslander-Reiten sequence in  $\mathcal{S}_2 \uparrow^{\ell-2}$  of type

$$0 \rightarrow T_2^{m-1,r-1} \uparrow^{\ell-2} \rightarrow T_2^{m,r-1} \uparrow^{\ell-2} \oplus T_2^{m-1,r} \uparrow^{\ell-2} \rightarrow T_2^{m,r} \uparrow^{\ell-2} \rightarrow 0$$

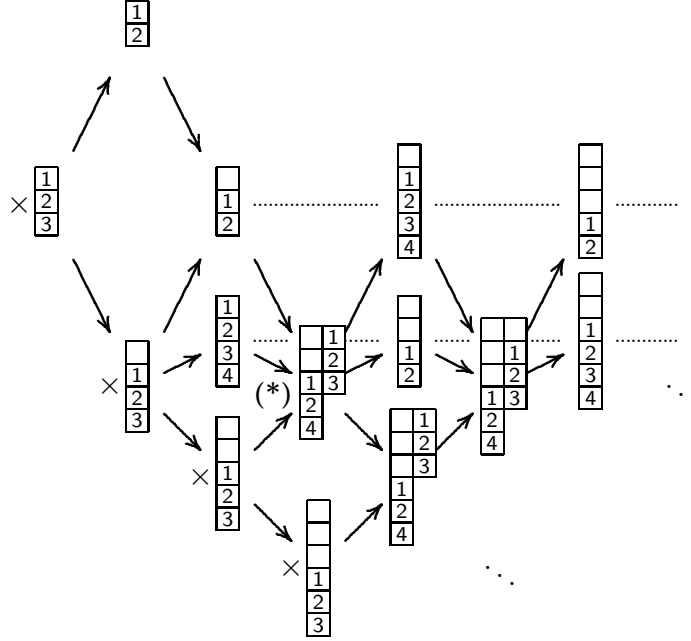
if  $m > r + 2$ , and in case if  $m = r + 2$  of type:

$$0 \rightarrow T_2^{m-1,r-1} \uparrow^{\ell-2} \rightarrow T_2^{m,r-1} \uparrow^{\ell-2} \oplus P_\ell^{m-1} \oplus P_{\ell-2}^r \rightarrow T_2^{m,r} \uparrow^{\ell-2} \rightarrow 0$$

✓

There is no assertion in the proposition about sink maps ending at pickets of type  $P_0^m \uparrow^{\ell-2}$  or  $P_1^m \uparrow^{\ell-2}$ . In fact, there exist sink maps in  $\mathcal{S}_2 \uparrow^{\ell-2}$  for the pickets of the form  $P_0^m \uparrow^{\ell-2}$ . As those maps will not be needed in the following, we leave it as an exercise for the reader to determine the corresponding Auslander-Reiten sequences. On the other hand, pickets of the form  $P_1 \uparrow^{\ell-2}$  are neither projective nor will they admit a sink map. In the example below, they are labeled with an  $\times$ .

*Example.* Here is the partial Auslander-Reiten quiver for  $\mathcal{S}_2 \uparrow^2$ :



In Example (3) in (4.6) we will use the Auslander-Reiten sequence ending at  $T_2^{5,3}\uparrow^2$  and labelled  $(*)$  to determine all objects with a  $\boxed{4_3}$  in the 5-th row of their Klein tableau.

**4.3. Approximations.** For  $E : (A \subset B)$  an embedding of  $p$ -modules, and  $\ell$  a natural number, put

$$E|^\ell = (A/p^\ell A \subset B/p^\ell A).$$

The following results follow immediately from the definition.

**LEMMA 8.** *The canonical map  $\pi : E \rightarrow E|^\ell$  is a minimal left approximation of  $E$  in  $\mathcal{S}_\ell$ .*

*Proof.* Every map  $E \rightarrow F$  with  $F \in \mathcal{S}_\ell$  factors over  $\pi$ , so  $\pi$  is a left approximation for  $E$  in  $\mathcal{S}_\ell$ . Since  $\pi$  is onto, any map  $u \in \text{End}(E|^\ell)$  which satisfies  $\pi = u\pi$  is an isomorphism, so  $\pi$  is in addition left minimal. ✓

**LEMMA 9.** *Suppose an embedding  $E \in \mathcal{S}$  has Klein tableau  $\Pi = [\gamma^0, \dots, \gamma^e; \varphi^2, \dots, \varphi^e]$ , and  $\ell$  is a natural number at most  $e$ . Then*

$$\Pi(E|^\ell) = [\gamma^0, \dots, \gamma^\ell; \varphi^2, \dots, \varphi^\ell] = \Pi|^\ell.$$

✓

We obtain from Lemma 6:

LEMMA 10. Suppose  $E$  has Klein tableau  $\Pi$  and  $k \leq \ell$ . The embeddings  $E|_{\downarrow_{\ell-k}}|^k$  and  $E|_{\downarrow_{\ell-k}}^\ell$  coincide as objects in  $\mathcal{S}$  and have Klein tableau

$$\Pi(E|_{\downarrow_{\ell-k}}|^k) = [\gamma^{\ell-k}, \gamma^{\ell-k+1}, \dots, \gamma^\ell; \varphi^{\ell-k+2}, \varphi^{\ell-k+3}, \dots, \varphi^\ell] = \Pi|_k^\ell.$$

In the case where  $k = 2$  we obtain

$$\Pi(E|_{\downarrow_{\ell-2}}|^2) = [\gamma^{\ell-2}, \gamma^{\ell-1}, \gamma^\ell; \varphi^\ell] = \Pi|_2^\ell.$$

✓

We write  $E|_k^\ell = E|_{\downarrow_{\ell-k}}^\ell = E|_{\downarrow_{\ell-k}}|^k$ . Combining this result with Proposition 2, we can interpret the number of boxes  $\boxed{\ell_r}$  in the Klein tableau of an embedding  $E : (A \subset B)$  in terms of multiplicities of summands in a subfactor of  $E$ .

COROLLARY 2. Let  $\Pi$  be the Klein tableau of an object  $E \in \mathcal{S}$ , and let  $2 \leq \ell \leq r+1 \leq m$  be positive integers. The number of boxes  $\boxed{\ell_r}$  in the  $m$ -th row of  $\Pi$  equals the multiplicity of the picket  $P_2^m$  (if  $r = m-1$ ) or the bipicket  $T_2^{m,r}$  (if  $r < m-1$ ) in the direct sum decomposition for

$$E|_2^\ell = (p^{\ell-2}A/p^\ell A \subset B/p^\ell A).$$

*Proof.* The number of boxes  $\boxed{\ell_r}$  in the  $m$ -th row of  $\Pi$  equals the number of boxes  $\boxed{2_r}$  in the  $m$ -th row of  $\Pi|_2^\ell$ . According to Proposition 2, for each symbol  $\boxed{2_r}$  there is an indecomposable summand of type  $P_2^m$  or  $T_2^{m,r}$  in the direct sum decomposition of the corresponding object  $E|_2^\ell$  in  $\mathcal{S}_2$ . ✓

We recover the corresponding result for LR-tableaux [10, Corollary 2] in the case where  $\ell > 1$ :

COROLLARY 3. Let  $\Gamma$  be the LR-tableau corresponding to an object  $E \in \mathcal{S}$ , and let  $1 \leq \ell \leq m$ . The number of boxes  $\boxed{\ell}$  in the  $m$ -th row of  $\Gamma$  equals the multiplicity of the picket  $P_1^m$  in the direct sum decomposition for  $E|_1^\ell$ .

*Proof.* Let  $\ell > 1$ . Since  $E|_1^\ell = E|_{\downarrow_1}^\ell$  and since

$$T_2^{m,r} \downarrow_1 = \begin{cases} P_1^m & \text{if } r = m-1 \\ P_1^m \oplus P_0^r & \text{if } r < m \end{cases}$$

the multiplicity of  $P_1^m$  as a direct summand of  $E|_1^\ell$  is the sum of the multiplicities of  $T_2^{m,r}$  in  $E|_2^\ell$  for  $r = 1, \dots, m-1$ . ✓

**4.4. Categorification.** Given an object  $E \in \mathcal{S}$ , we can interpret the entries in the Klein tableau for  $E$  in terms of homomorphisms in  $\mathcal{S}$  and in terms of the decomposition of subfactors of  $E$ .



We restate Theorem 2 from the introduction. The numbers  $m, r, \ell$  are chosen in such a way that all bipickets  $T_2^{m,r} \uparrow^{\ell-2}$  and all pickets  $P_\ell^m$  with  $\ell \geq 2$  are included.

**THEOREM 4.** *For  $E \in \mathcal{S}$  and  $2 \leq \ell \leq r+1 \leq m$ , the following numbers are equal.*

- (1) *The number of boxes  $\boxed{\ell_r}$  in the  $m$ -th row in the Klein tableau for  $E$ .*
- (2) *The multiplicity of  $T_2^{m,r}$  as a direct summand of*

$$E|_2^\ell = (p^{\ell-2}A/p^\ell A \subset B/p^\ell A).$$

- (3) *The  $k$ -dimension of*

$$\frac{\text{Hom}_{\mathcal{S}}(E, T_2^{m,r} \uparrow^{\ell-2})}{\text{Im Hom}_{\mathcal{S}}(E, v_2^{m,r} \uparrow^{\ell-2})}.$$

*Remark.* (1) In the theorem,  $v_2^{m,r}$  is the sink map for  $T_2^{m,r}$  in the category  $\mathcal{S}_2$  as introduced in (2.3). The lifting  $v_2^{m,r} \uparrow^{\ell-2}$  may not be the sink map for  $T_2^{m,r} \uparrow^{\ell-2}$ , but its minimal version  $v$  is (Proposition 4). In particular,  $\text{Im Hom}(E, v_2^{m,r} \uparrow^{\ell-2}) = \text{Im Hom}(E, v)$  consists of all maps which factor over the sink map for  $T_2^{m,r} \uparrow^{\ell-2}$  in the category  $\mathcal{S}_2 \uparrow^{\ell-2}$ .

- (2) The above theorem covers all entries in the Klein tableau with the exception of the  $\boxed{1}$ 's. Those entries occur also in the underlying LR-tableau and are dealt with by [10, Theorem 1]: The multiplicity of  $\boxed{\ell}$  in the  $m$ -th row of the LR-tableau for  $E$  equals the multiplicity of  $P_1^m$  as a direct summand of  $E|_1^\ell$  and also equals the  $k$ -dimension of

$$\frac{\text{Hom}(E, P_\ell^m)}{\text{Im Hom}(E, u_\ell^m)}$$

where  $u_1^m$  is the sink map for  $P_1^m$  in the category  $\mathcal{S}_1$  and  $u_\ell^m = u_1^m \uparrow^{\ell-1}$ .

*Proof.* According to Corollary 2, the number of boxes labelled  $\boxed{\ell_r}$  in row  $m$  is equal to the multiplicity of  $T_2^{m,r}$  as a direct summand of  $E|_2^\ell = E|_{\ell-2}^\ell|^2$ . In the category  $\mathcal{S}_2$ , this multiplicity is measured as the dimension of the contravariant defect given by the Auslander-Reiten sequence ending at  $T_2^{m,r}$ . This dimension is equal to

$$\dim_k \frac{\text{Hom}(E|_{\ell-2}^\ell|^2, T_2^{m,r})}{\text{Im Hom}(E|_{\ell-2}^\ell|^2, v_2^{m,r})}.$$

Using Lemma 8, this number is equal to

$$\dim_k \frac{\operatorname{Hom}(E|_{\ell-2}, T_2^{m,r})}{\operatorname{Im} \operatorname{Hom}(E|_{\ell-2}, v_2^{m,r})}.$$

Now we apply the adjoint isomorphism from Lemma 5 to obtain equality with the expression in the theorem:

$$\dim_k \frac{\operatorname{Hom}(E, T_2^{m,r} \uparrow^{\ell-2})}{\operatorname{Im} \operatorname{Hom}(E, v_2^{m,r} \uparrow^{\ell-2})}.$$

✓

As a consequence of the theorem, we can read off from the Klein tableau of an embedding the length of the module of homomorphisms into a bipicket. The key step is the reduction to the corresponding situation for pickets:

**COROLLARY 4.** *Suppose  $E \in \mathcal{S}$  has Klein tableau  $\Pi$ . For integers  $1 \leq r \leq m-2$ , let*

$$b = \# \{ \text{symbols } \boxed{2_u} \text{ in row } v \text{ in } \Pi \mid r+2 \leq v \leq m, 1 \leq u \leq r \}$$

*Then*

$$\begin{aligned} \operatorname{len} \operatorname{Hom}_{\mathcal{S}}(E, T_2^{m,r}) &= b + \operatorname{len} \operatorname{Hom}_{\mathcal{S}}(E, P_2^{r+1} \oplus P_0^r \oplus P_1^m) \\ &\quad - \operatorname{len} \operatorname{Hom}_{\mathcal{S}}(E, P_1^{r+1}). \end{aligned}$$

The length of the homomorphism group into a picket  $P_\ell^m$  can be read off from the LR-tableau  $\Gamma = [\gamma^0, \dots, \gamma^e]$  of the module  $E : (A \subset B)$ :

$$\operatorname{len} \operatorname{Hom}_{\mathcal{S}}(E, P_\ell^m) = \sum_{i=1}^m (\gamma^\ell)_i'$$

(For this recall that  $\operatorname{Hom}_{\mathcal{S}}(E, P_\ell^m) = \operatorname{Hom}_R(B/p^\ell A, P^m)$  as discussed in [10], and note that  $B/p^\ell A = \bigoplus_j P^{\gamma_j^\ell}$  and that  $\operatorname{len} \operatorname{Hom}_R(P^s, P^m) = \min\{s, m\}$ .)

*Proof of the Corollary.* Since  $\operatorname{Hom}(E, F) = \operatorname{Hom}(E|^2, F)$  for  $F \in \mathcal{S}_2$  we may assume that  $E \in \mathcal{S}_2$ . We first consider the case where  $E$  has no summand of type  $T_2^{v,u}$ ; then the functor  $\operatorname{Hom}_{\mathcal{S}}(E, -)$  is exact when applied to the Auslander-Reiten sequence ending at  $T_2^{v,u}$ .

In the computation below we proceed along the diagonals in the Auslander-Reiten quiver for  $\mathcal{S}_2$ , going first from  $T_2^{m,r}$  to  $T_2^{m,1}$ , then from  $T_2^{m-1,r}$  to  $T_2^{m-1,1}$ , etc. and finally from  $T_2^{r+2,r}$  to  $T_2^{r+2,1}$ . In each step

we replace the term  $(E, T_2^{v,u})$ , which is the length of the third term of the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(E, X) &\rightarrow \text{Hom}(E, Y) \xrightarrow{\text{Hom}(E, g^{v,u})} \text{Hom}(E, T_2^{v,u}) \\ &\rightarrow \text{Cok Hom}(E, g^{v,u}) \rightarrow 0 \end{aligned}$$

given by applying the functor  $\text{Hom}(E, -)$  to the Auslander-Reiten sequence ending at  $T_2^{v,u}$  by

$$\text{len Hom}(E, Y) - \text{len Hom}(E, X).$$

$$\begin{aligned} (E, T_2^{m,r}) &= (E, T_2^{m,r-1}) + (E, T_2^{m-1,r}) - (E, T_2^{m-1,r-1}) \\ &= (E, T_2^{m,r-2}) + (E, T_2^{m-1,r}) - (E, T_2^{m-1,r-2}) \\ &= \cdots = (E, T_2^{m,1}) + (E, T_2^{m-1,r}) - (E, T_2^{m-1,1}) \\ &= (E, P_1^m) + (E, T_2^{m-1,r}) - (E, P_1^{m-1}) \\ &= \cdots = (E, P_1^m) + (E, T_2^{r+2,r}) - (E, P_1^{r+2}) \\ &= (E, P_1^m) + (E, P_2^{r+1}) + (E, P_0^r) + (E, T_2^{r+2,r-1}) \\ &\quad - (E, T_2^{r+1,r-1}) - (E, P_1^{r+2}) \\ &= \cdots = (E, P_1^m) + (E, P_2^{r+1}) + (E, P_0^r) - (E, P_1^{r+1}) \end{aligned}$$

Here  $(E, Y)$  denotes the length  $\text{len Hom}_{\mathcal{S}}(X, Y)$ , This shows the claim in case  $b = 0$ .

Let us now consider an arbitrary embedding  $E$ . Then for a given pair  $(v, u)$ , the above replacement of  $(E, T_2^{v,u})$  by  $(E, Y) - (E, X)$  omits the term  $\text{len Cok}(E, g^{v,u})$  which according to Theorem 2 counts the number of boxes  $\boxed{2_u}$  in the  $v$ -th row in the Klein tableau for  $E$ . For each pair  $(v, u)$  where  $r + 2 \leq v \leq m$  and  $1 \leq u \leq r$  one such omission occurs; together they sum up to yield the extra summand  $b$  in the formula in the statement of the result.  $\checkmark$

**4.5. The location of symbols in the category  $\mathcal{S}(n)$ .** The following result may help to detect the objects in an Auslander-Reiten quiver which have a certain entry in their Klein tableau. Suppose  $E \in \mathcal{S}(n)$  has a symbol  $\boxed{\ell_r}$  in the  $m$ -th row of its Klein tableau (so  $\ell, r, m, n$  satisfy the inequalities  $2 \leq \ell \leq r + 1 \leq m \leq n$ ). We have seen in Theorem 4 that there is a map  $g : E \rightarrow Z$  where  $Z = T_2^{m,r} \uparrow^{\ell-2}$  which does not factor through the sink map  $v$  for  $Z$  in  $\mathcal{S}_2 \uparrow^{\ell-2}$ . In the next statement we show that there is a corresponding object  $C$  depending only on  $Z$  and  $n$ , and a map  $f : C \rightarrow E$  such that  $gf$  does not factor through  $v$ . In this sense,  $E$  is “in between”  $C$  and  $Z$ .

THEOREM 5. *Suppose that the integers  $\ell, r, m, n$  satisfy*

$$2 \leq \ell \leq r + 1 \leq m \leq n.$$

*Let  $C \in \mathcal{S}(n)$  be defined as follows: The picket or bipicket  $Z = T_2^{m,r} \uparrow^{\ell-2}$  is either projective (i.e.  $Z = P_\ell^\ell$ ) in  $\mathcal{S}_2 \uparrow^{\ell-2}$  with sink map  $P_{\ell-1}^\ell \xrightarrow{v} P_\ell^\ell$  in which case we put  $C = P_n^n$ ; or else  $Z$  occurs as end term of an Auslander-Reiten sequence  $0 \rightarrow X \rightarrow Y \xrightarrow{v} Z \rightarrow 0$  in  $\mathcal{S}_2 \uparrow^{\ell-2}$ , in which case we put  $C = \tau_{\mathcal{S}(n)}^{-1} X$ .*

(1) *The  $R$ -module*

$$\frac{\text{Hom}_{\mathcal{S}}(C, Z)}{\text{Im Hom}_{\mathcal{S}}(C, v)}$$

*is a one dimensional  $k$ -vector space.*

(2) *For  $E \in \mathcal{S}(n)$ , the map given by composition,*

$$\frac{\text{Hom}(E, Z)}{\text{Im Hom}(E, v)} \times \text{Hom}(C, E) \longrightarrow \frac{\text{Hom}(C, Z)}{\text{Im Hom}(C, v)}$$

*is left non-degenerate.*

The corresponding result about the entries in the LR-tableau is [10, Proposition 1].

*Proof.* For the first statement we verify in each of five cases that the Klein tableau for  $C$  contains exactly one symbol  $\boxed{\ell_r}$  in the  $m$ -th row. This implies by Theorem 4 that the  $k$ -vector space  $\frac{\text{Hom}_{\mathcal{S}}(C, Z)}{\text{Im Hom}_{\mathcal{S}}(C, v)}$  has dimension one.

(a) In the first case where  $\ell = r + 1 = m$ , the module

$$Z = T_2^{m,r} \uparrow^{\ell-2} = P_\ell^\ell$$

is the indecomposable projective object in  $\mathcal{S}_2 \uparrow^{\ell-2}$ ; the corresponding module  $C = P_n^n$  has the following Klein tableau:

$$P_n^n : \begin{bmatrix} \boxed{1} \\ \vdots \\ \boxed{n} \end{bmatrix}$$

In particular, the  $m$ -th row has entry  $\boxed{\ell_r}$  where  $\ell = m$  and  $r = m - 1$ .

(b) In this case we assume that  $\ell < r + 1 = m$ , so we are dealing with a nonprojective picket  $Z = T_2^{m,r} \uparrow^{\ell-2} = P_\ell^m$ . We have seen in (4.2) that  $\tau_{\mathcal{S}_2 \uparrow^{\ell-2}} P_\ell^m = P_{\ell-2}^{m-2}$ . Here and in the remaining cases we use [7, Theorem 5.2, see also Lemma 1.2 (3)] to compute

the inverse of the Auslander-Reiten translation in  $\mathcal{S}(n)$  as the kernel of the minimal epimorphism representing the embedding:

$$\begin{aligned}
C &= \tau_{\mathcal{S}(n)}^{-1} P_{\ell-2}^{m-2} \\
&= \ker \text{mepi}(P^{\ell-2} \rightarrow P^{m-2}, y \mapsto p^{m-\ell} y) \\
&= \ker(P^n \oplus P^{\ell-2} \rightarrow P^{m-2}, (x, y) \mapsto x + p^{m-\ell} y) \\
&= (P^{n+\ell-m} \rightarrow P^n \oplus P^{\ell-2}, u \mapsto (p^{m-\ell} u, -u)) \\
&= ((p^{m-\ell}, -1)) \subset P^n \oplus P^{\ell-2}
\end{aligned}$$

This embedding has the following Klein tableau:

$$C : \begin{array}{c} \left. \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline \ell'' \\ \hline \end{array} \right\} m-2 \\ \begin{array}{|c|} \hline \ell' \\ \hline \ell \\ \hline \vdots \\ \hline t \\ \hline \end{array} \end{array}$$

Here we abbreviate  $\ell' = \ell - 1$ ,  $\ell'' = \ell - 2$  and put  $t = n + \ell - m$ . We see that the  $m$ -th row consists of the symbol  $\boxed{\ell_r}$  where  $r = m - 1$ .

- (c) Here we consider the case where  $\ell = r + 1 < m$ . Then the translate of the bipicket  $Z = T_2^{m,r} \uparrow^{\ell-2}$  is the picket  $\tau_{\mathcal{S}_2 \uparrow^{\ell-2}} Z = P_{\ell-1}^{m-1}$ . As above,  $C = \tau_{\mathcal{S}(n)}^{-1} P_{\ell-1}^{m-1}$  is computed as

$$C = ((p^{m-\ell}, -1)) \subset P^n \oplus P^{\ell-1};$$

its Klein tableau

$$C : \begin{array}{c} \left. \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline \ell' \\ \hline \end{array} \right\} m-1 \\ \begin{array}{|c|} \hline \ell \\ \hline \vdots \\ \hline t \\ \hline \end{array} \end{array}$$

has the symbol  $\boxed{\ell_r}$  where  $r = \ell - 1$  in the  $m$ -th row ( $t = n + \ell - m$ ).

- (d<sub>1</sub>) We assume  $\ell + 1 = r + 1 < m$ . The translate of the bipicket  $Z = T_2^{m,r} \uparrow^{\ell-2}$  is the bipicket  $\tau_{\mathcal{S}_2 \uparrow^{\ell-2}} Z = T_2^{m-1,r-1} \uparrow^{\ell-2}$ . Note that here — unlike in the next case — the factor of the embedding

defining this module is cyclic. We compute

$$\begin{aligned}
C &= \tau_{S(n)}^{-1} T_2^{m-1, r-1} \uparrow^{\ell-2} \\
&= \ker \text{mepi}(P^\ell \oplus P^{\ell-2} \rightarrow P^{m-1} \oplus P^{r-1}, \\
&\quad (y, z) \mapsto (p^{m-1-\ell} y, p^{r-\ell} y + p^{r-\ell+1} z)) \\
&= \ker \text{mepi}(P^\ell \oplus P^{\ell-2} \rightarrow P^{m-1} \oplus P^{r-1}, \\
&\quad (y, z) \mapsto (p^{m-1-\ell} y, y + pz)) \\
&= \ker(P^n \oplus P^\ell \oplus P^{\ell-2} \rightarrow P^{m-1} \oplus P^{r-1}, \\
&\quad (x, y, z) \mapsto (x + p^{m-1-\ell} y, y + pz)) \\
&= (P^{n+\ell-m} \rightarrow P^n \oplus P^\ell \oplus P^{\ell-2}, u \mapsto (p^{m-\ell} u, -pu, u)) \\
&= ((p^{m-\ell}, -p, 1)) \subset P^n \oplus P^\ell \oplus P^{\ell-2}
\end{aligned}$$

The Klein tableau is as follows:

$$C : \quad \left\{ \begin{array}{c} \boxed{1} \\ \vdots \\ \boxed{\ell'} \\ \boxed{\ell'} \end{array} \right\}^\ell$$

$\begin{array}{c} \boxed{\ell} \\ \vdots \\ \boxed{t} \end{array}$

Namely, the sequence of radical layers of the elements  $p^i u$  in the total space is  $1, 2, \dots, \ell-2, \ell, m, m+1, \dots, t = n + \ell - m$ . So the  $m$ -th row consists of the symbol  $\boxed{\ell_r}$  where  $r = \ell$ .

- (d<sub>2</sub>) Finally we consider the case where  $\ell+1 < r+1 < m$ . The translate of  $Z = T_2^{m,r} \uparrow^{\ell-2}$  is the bipicket  $\tau_{S_2 \uparrow^{\ell-2}} Z = T_2^{m-1, r-1} \uparrow^{\ell-2}$ . We compute

$$\begin{aligned}
C &= \tau_{S(n)}^{-1} T_2^{m-1, r-1} \uparrow^{\ell-2} \\
&= \ker \text{mepi}(P^\ell \oplus P^{\ell-2} \rightarrow P^{m-1} \oplus P^{r-1}, \\
&\quad (y, z) \mapsto (p^{m-1-\ell} y, p^{r-\ell} y + p^{r-\ell+1} z)) \\
&= \ker(P^n \oplus P^n \oplus P^\ell \oplus P^{\ell-2} \rightarrow P^{m-1} \oplus P^{r-1}, \\
&\quad (w, x, y, z) \mapsto \\
&\quad (w + p^{m-1-r} x + p^{m-1-\ell} y, x + p^{r-\ell} y + p^{r-\ell+1} z)) \\
&= (P^{n-m+\ell} \oplus P^{n-r+\ell} \rightarrow P^n \oplus P^n \oplus P^\ell \oplus P^{\ell-2}, \\
&\quad (u, v) \mapsto (p^{m-\ell} u, p^{r-\ell} v, -pu - v, u))
\end{aligned}$$

The Klein tableau is as follows:

$$C : \begin{array}{c} \left\{ \begin{array}{c} \boxed{1} \boxed{1} \\ \vdots \\ \boxed{\ell'} \boxed{\ell''} \\ \boxed{\ell'} \\ \boxed{\ell} \end{array} \right\}^{r-1} \\ \left\{ \begin{array}{c} \boxed{\ell'} \\ \boxed{\ell^*} \\ \boxed{\ell} \\ \vdots \\ \boxed{\ell^*} \\ \vdots \\ \boxed{t} \boxed{t'} \end{array} \right\}^{m-1} \end{array}$$

Here  $\ell^* = \ell + 1$ ,  $t = n - m + \ell$  and  $t' = n - r + \ell$ . We determine the two symbols corresponding to the labels  $\ell$ . Write  $C$  as the embedding  $(A \subset B)$  and verify that the types of  $B/p^{\ell-2}A$ ,  $B/p^{\ell-1}A$  and  $B/p^\ell A$  are  $(m-1, r-1, \ell-2, \ell-2)$ ;  $(m-1, r, \ell-1, \ell-2)$ ; and  $(m, r, \ell, \ell-2)$ , respectively. Thus, the two labels  $\ell$  occur in rows  $m$  and  $\ell$ , while the two labels  $\ell-1$  occur in rows  $r$  and  $\ell-1$ . Since  $r > \ell$ , the subscript  $r$  remains for the label  $\ell$  in row  $m$ . So the two symbols  $\boxed{\ell_r}$  and  $\boxed{\ell_{\ell'}}$  occur in rows  $m$  and  $\ell$ , respectively.

For the proof of the second statement, we first consider the case where  $Z = P_\ell^\ell$  is projective in  $\mathcal{S}_2 \uparrow^{\ell-2}$ . Let  $\pi : P_n^n \rightarrow P_\ell^\ell$  be the canonical map. Suppose a map  $f : E \rightarrow P_\ell^\ell$  with  $E \in \mathcal{S}(n)$  does not factor through the sink map  $P_{\ell-1}^\ell \rightarrow P_\ell^\ell$ ; then  $f$  is an epimorphism. Recall that  $P_n^n$  is a projective object in the abelian category  $\mathcal{H}(n)$  of all maps between  $R/(p^n)$ -modules; and that  $\mathcal{S}(n) \subset \mathcal{H}(n)$  is a full subcategory. It follows that  $\pi$  factors through the epimorphism  $f$ .

For the case where  $Z$  is nonprojective, we adapt the second part of the proof of [10, Proposition 2]: Suppose that  $E \in \mathcal{S}(n)$  is given. Let  $\mathcal{A} : 0 \rightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow 0$  be the Auslander-Reiten sequence in  $\mathcal{S}_2 \uparrow^{\ell-2}$  ending at  $Z$ , as given in Proposition 4. Let

$$\mathcal{E} : 0 \rightarrow X \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

be the Auslander-Reiten sequence in  $\mathcal{S}(n)$  starting at  $X$ , its end term is  $C$ . Since  $\mathcal{A}$  is nonsplit, there are maps  $h' : B \rightarrow Y$ ,  $h : C \rightarrow Z$  which

make the upper part of the following diagram commutative.

$$\begin{array}{ccccccccc}
\mathcal{E} : & 0 & \longrightarrow & X & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
& & & \parallel & & \downarrow h' & & \downarrow h & & \\
\mathcal{A} : & 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0 \\
& & & \parallel & & \uparrow p' & & \uparrow p & & \\
p\mathcal{A} : & 0 & \longrightarrow & X & \xrightarrow{s} & L & \xrightarrow{t} & E & \longrightarrow & 0
\end{array}$$

In order to show that the bilinear form given by composition

$$\frac{\text{Hom}(E, Z)}{\text{Im Hom}(E, v)} \times \text{Hom}(C, E) \longrightarrow \frac{\text{Hom}(C, Z)}{\text{Im Hom}(C, v)}$$

is left non-degenerate, let  $p : E \rightarrow Z$  be a map which does not factor through  $v$ . We will construct  $q : C \rightarrow E$  such that  $pq$  does not factor through  $v$ . Since  $p$  does not factor through  $v$ , the induced sequence at the bottom of the above diagram does not split. Hence the map  $s$  factors through  $f$ : There is a map  $q' : B \rightarrow L$  such that  $s = q'f$ . Let  $q : C \rightarrow E$  be the cokernel map, so  $qg = tq'$ . Then  $pqg = ptq' = vp'q'$ . Since  $p'q'f = p's = u = h'f$ , there exists  $z : C \rightarrow Y$  such that  $zg = p'q' - h'$ . So  $pqg = vp'q' = v(zg + h') = (vz + h)g$  and since  $g$  is onto,  $pq = vz + h$ . Since  $\mathcal{E}$  is not split exact,  $h$  does not factor through  $v$ , and hence  $pq$  does not factor through  $v$ .  $\checkmark$

**4.6. Examples.** (1) The diagram below represents the Auslander-Reiten quiver for the category  $\mathcal{S}(5)$ , which has the largest number of indecomposable objects among all representation finite categories of type  $\mathcal{S}(n)$ . We refer to [8, Section 6.5] for a detailed description of the category  $\mathcal{S}(5)$  and its objects. In the diagram, each object is represented by its LR-tableau. Note that each LR-tableau of an indecomposable object in  $\mathcal{S}(5)$  can be refined uniquely to a Klein tableau, so we may omit the subscripts of the labels.

Let us consider as in [10] the encircled region  $\mathcal{R}$  consisting of all indecomposables which have an entry  $\boxed{2}$  in their 4th row. Note that the two “eyes” are not part of the region  $\mathcal{R}$ . Let

$$0 \rightarrow X \rightarrow Y_1 \oplus Y_2 \xrightarrow{v} Z \rightarrow 0$$

be the Auslander-Reiten sequence in the category  $\mathcal{S}_1 \uparrow^1$  ending at  $Z = P_2^4$ ; here  $X = P_1^3$ ,  $Y_1 = P_1^4$  and  $Y_2 = P_2^3$ . The modules  $E$  in  $\mathcal{R}$  are characterized by the existence of a homomorphism in  $\text{Hom}(E, Z)$  which does not factor over  $v$  [10, Theorem 1], and also by the existence of a



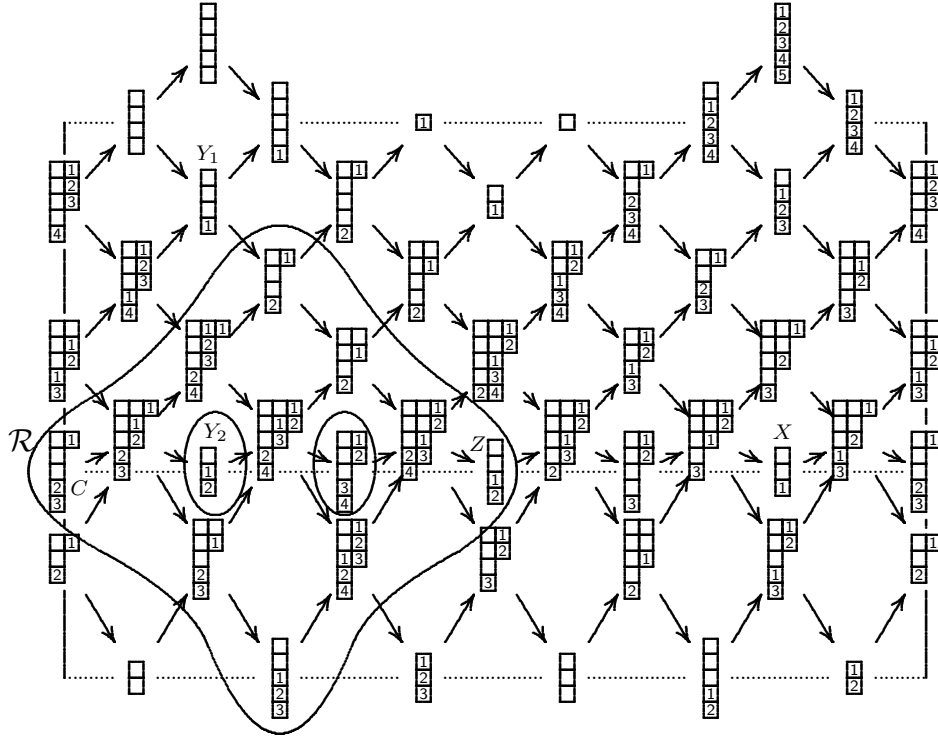
map  $C = \tau_{\mathcal{S}(5)}^{-1}X \rightarrow Z$ , not in  $\text{Im Hom}(C, v)$ , which factors through  $E$  [10, Proposition 2].

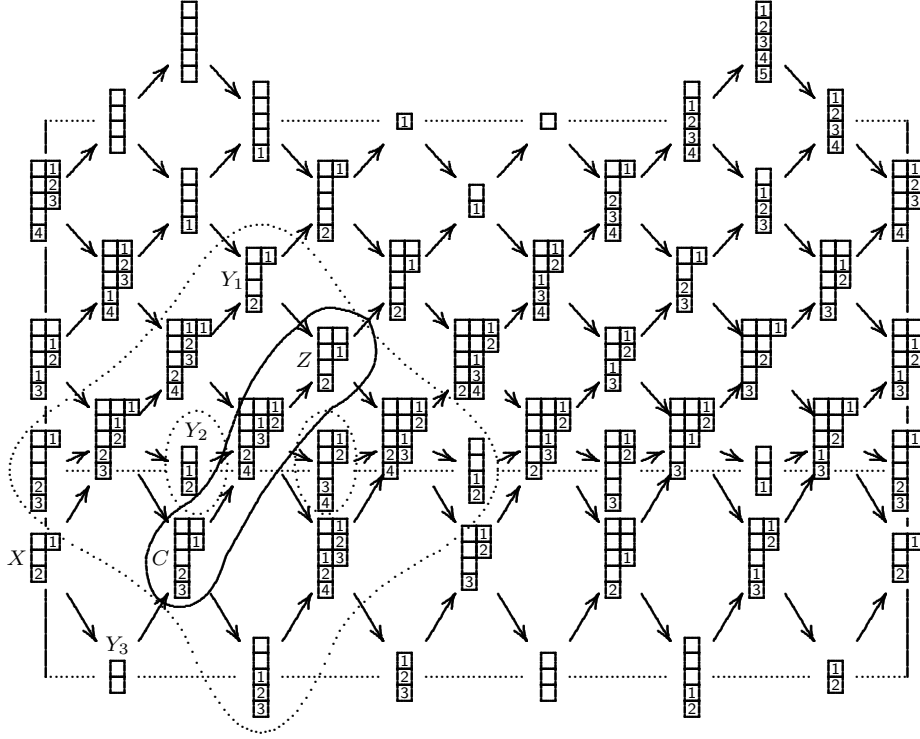
(2) As Klein tableaux are refinements of LR-tableaux, the entries in the Klein tableau provide an even finer selection of indecomposable objects. When adding subscripts to the entries in the LR-tableau, the entry  $\boxed{2}$  in the 4th row may become one of the symbols  $\boxed{2_1}$ ,  $\boxed{2_2}$ , or  $\boxed{2_3}$ . In fact, the three modules along the middle diagonal, which are encircled in the second diagram, all have a box  $\boxed{2_2}$  in the 4th row of their Klein tableau. The modules above them will carry a  $\boxed{2_1}$ , and those below them a  $\boxed{2_3}$ .

Returning to the modules which have a  $\boxed{2_2}$  in the 4th row of their Klein tableau, we consider the Auslander-Reiten sequence in  $\mathcal{S}_2$  ending at the bipicket  $Z = T_2^{4,2}$ :

$$0 \rightarrow X \rightarrow Y_1 \oplus Y_2 \oplus Y_3 \xrightarrow{v} Z \rightarrow 0$$

Here,  $X = T_2^{3,1}$ ,  $Y_1 = T_2^{4,1}$ ,  $Y_2 = P_2^3$  and  $Y_3 = P_0^2$ . According to Theorem 2, each of the modules  $E$  in the marked region will admit a map  $E \rightarrow Z$  which does not factor through  $v$ . And according to Theorem 5, there is a map  $C = \tau_{\mathcal{S}(5)}^{-1}X \rightarrow Z$ , not in  $\text{Im Hom}(C, v)$ , which factors through  $E$ .



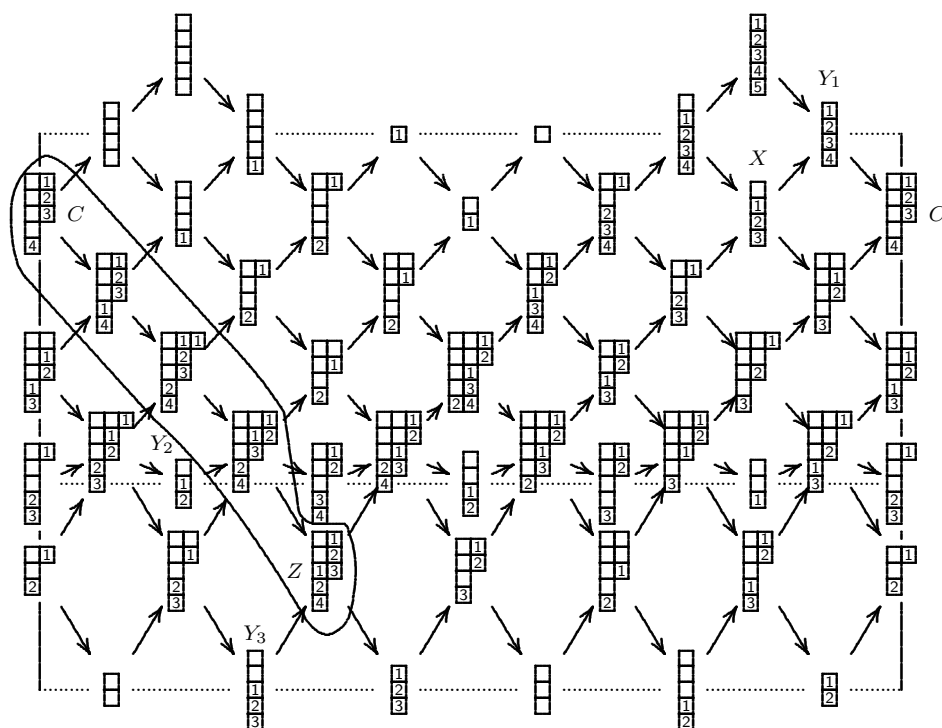


(3) In order to determine the modules which carry a symbol  $\boxed{\ell_r}$  where  $\ell > 2$ , the Auslander-Reiten sequences from (4.2) can be used. We consider as example the modules which have a  $\boxed{4_3}$  in row 5. They are encircled in the third diagram.

Consider the module  $Z = T_2^{5,3}\uparrow^2$  on the right hand side of the marked region. It occurs as the end term of the Auslander-Reiten sequence in  $\mathcal{S}_2\uparrow^2$ :

$$0 \rightarrow X \rightarrow Y_1 \oplus Y_2 \oplus Y_3 \xrightarrow{v} Z \rightarrow 0,$$

here  $X = P_3^4$ ,  $Y_1 = P_4^4$ ,  $Y_2 = P_2^3$ , and  $Y_3 = P_3^5$ ; this sequence is pictured on the left hand side in the partial Auslander-Reiten quiver in (4.2). As predicted by Theorem 2, the modules in the region are those which admit a map into  $Z$  which does not factor over  $v$ . Putting  $C = \tau_{\mathcal{S}(5)}^{-1}(X)$ , the modules in this region are those  $E$  for which there is a map  $C \rightarrow Z$ , not in  $\text{Im Hom}(C, v)$ , which factors through  $E$  (see Theorem 5).



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